Unit - 2

Beta and Gamma functions

Definitions :

1) The Beta function :

The Beta function $\beta(m,n)$ is defined by the difinite integral,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

where m and n are positive.

2) Gamma function :

The gamma function Γn is defined by the definite integral.

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$$

where n is positive. The restriction on n is necessary to avoid divergence of the integral.

Symmetry property of Beta function :

To show that $\beta(m,n) = \beta(n,m)$

By definition,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \longrightarrow (1)$$

Put $x = 1-y$, $dx = -dy$

$x = 0 ; 0 = 1-y$
$\therefore y = 1$

$x = 1 ; 1 = 1-y$
$1-y = 1$
$1 = 1+y$
$y = 1-1$
$y = 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} [1-(1-y)]^{n-1} [-dy]$$

$$= -\int_1^0 (1-y)^{m-1} (y)^{n-1} dy$$

$$= \int_0^1 (1-y)^{m-1} \cdot (y)^{n-1} dy$$

$$= \int_0^1 x^{n-1} \cdot (1-x)^{m-1} dx$$

$$\therefore \beta(m,n) = \beta(n,m) \longrightarrow (2)$$

Hence Proved //

2) Evaluation of Beta function :

Definition

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\, dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases}$$

Integrating by parts keeping $(1-x)^{n-1}$ as first function, we have.

we know that,

$$\int u\, dv = uv - \int v\, du$$

$$= \left[(1-x)^{n-1} \cdot \left(x^m/m\right)\right]_0^1 - \int \left(\frac{x^m}{m}\right)\left(-(n-1)(1-x)^{n-2}\right) dx$$

nx^{n-1}

$\int x^n = \frac{x^{n+1}}{n+1}$

sub limit value

$$= \left[(1-1)^{n-1} \cdot \frac{(1)^m}{m} - (1-0)^{n-1}\left(0^m/m\right)\right] + \int \frac{x^m}{m}(n-1)(1-x)^{n-2}\, dx$$

$$= \frac{(n-1)}{m}\int_0^1 x^m \cdot (1-x)^{n-2}\, dx$$

$$= \frac{(n-1)}{m}\int_0^1 (1-x)^{n-2} \cdot x^m\, dx$$

$u = (1-x)^{n-1}$

$\boxed{du = n-1\,(1-x)^{n-1-1}\, dx}$

$dv = x^{m-1}$

$V = \int x^{m-1}\, dx$

$= \frac{x^{m-1+1}}{m-1+1}$

$\therefore V = x^m/m$

Integrating again by parts, we get

$$\beta(m,n) = \frac{(n-1)(n-2)}{m(m+1)} \int_0^1 (1-x)^{n-3} x^{m+1}\, dx$$

Continuing the process of integrating by parts and assuming that n is a positive integer.

$$\beta(m,n) = \frac{(n-1)(n-2)\ldots\ldots 2\cdot 1}{m(m+1)\ldots\ldots(m+n-2)} \int_0^1 x^{m+n-2}\, dx$$

$$= \frac{(n-1)(n-2)\ldots\ldots 2\cdot 1}{m(m+1)\ldots\ldots(m+n-2)} \left[\frac{x^{m+n-2+1}}{m+n-2+1}\right]_0^1$$

$$\beta(m,n) = \frac{(n-1)!}{m(m+1)\ldots\ldots(m+n-2)(m+n-1)} \longrightarrow (2)$$

Again if m is also positive integer, then

$$\beta(m,n) = \frac{(n-1)!\,(m+1)!}{(n+m-1)!}$$

Transformation of Beta function (or) other forms of Beta function:

By definition:

$$\beta(m,n) = \int_0^1 x^{(m-1)} (1-x)^{n-1}\, dx \longrightarrow (1)$$

a) Put

$$x = \frac{y}{1+y}, \quad dx = \frac{dy}{(1+y)^2}$$

$$1-x = 1 - y/_{1+y}$$

$$(1-x) = \frac{1+y-y}{1+y} \Rightarrow \frac{1}{1+y}$$

$x = 0$

$x = \frac{y}{1+y}$

$0 = \frac{y}{1+y}$

$y = 0$

$x = 1$

$1 = \frac{y}{1+y}$

$1+y = y$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}} \cdot dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy \longrightarrow (2)$$

b) We have, $\beta(m,n) = \beta(n,m)$

$$\therefore \beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy \longrightarrow (3)$$

Eq (3) can be obtained directly from eqn (1) by substituting $x = \frac{1}{1+y}$.

Other form of gamma function

By definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \longrightarrow (1)$$

a) Putting $x = \lambda y$, $dx = \lambda\, dy$ in eqn (1)

we get

$$\Gamma n = \int_0^\infty e^{-\lambda y} (+\lambda y)^{n-1} \lambda' dy$$

base eqn $(\lambda y)^{n-1} \cdot \lambda^1$

$$= \int_0^\infty e^{-\lambda y} \lambda^n \cdot y^{n-1}\, dy$$

$\lambda^{n-\cancel{1}+\cancel{1}} \cdot y^{n-1}$

$$\Gamma n = \lambda^n \int_0^\infty e^{-\lambda y} \cdot y^{(n-1)}\, dy$$

$$\Gamma n = \int_0^\infty e^{-\lambda y} y^{(n-1)}\, dy = \frac{\Gamma n}{\lambda^n} \longrightarrow (2)$$

b) Put $\boxed{e^{-x} = y}$

$$\frac{1}{e^x} = y$$

$$\frac{1}{y} = e^x$$

$\frac{1}{x} = \log x + c$

$e^x = e^x + c$

$$\boxed{\log\left(\frac{1}{y}\right) = x}$$

dub x^0

$$dx = -\frac{dy}{y}$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \qquad e^{-x} = y, \quad x = \log\frac{1}{y}, \quad dx = -\frac{dy}{y}$$

$$\Gamma n = -\int_1^0 \cancel{y} \cdot \log\left(\frac{1}{y}\right)^{n-1} \cdot \frac{dy}{\cancel{y}}$$

$$= -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} dy$$

$$\Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \longrightarrow 3$$

c) $\Gamma n = \int_0^\infty e^{-x} x^{(n-1)} dx$

Put

$$x^n = y$$

$$x = y^{1/n}$$

$$dx = \frac{1}{n} y^{\frac{1}{n}-1} \cdot dy$$

$$dx = \frac{1}{n} y^{\frac{(1-n)}{n}} dy$$

sub in definion eqn 1

$$= \int_0^\infty e^{-(y^{1/n})} \left(y^{1/n}\right)^{(n-1)} \cdot \frac{1}{n}\left(y^{\frac{1-n}{n}}\right) dy$$

$$= \frac{1}{n}\int_0^\infty e^{-(y^{1/n})} \left(y^{1/n}\right)^{(n-1)} \cdot \left(y^{\frac{1-n}{n}}\right) dy$$

$$= \frac{1}{n}\int_0^\infty e^{-y^{1/n}} \cancel{y^{\frac{n-1}{n}}}\, \cancel{y^{\frac{1-n}{n}}}\, dy$$

$$\Gamma n = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\, dy \longrightarrow (4)$$

Relation Between Beta and Gamma functions:

Imp Imp

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

By definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \longrightarrow (1)$$

Put

$$x = \lambda y$$

$$dx = \lambda\, dy \quad \text{in eqn (1)}$$

$! = \frac{1\times 2}{1\times 2}$

gamma = factor

we get

$$\Gamma n = \int_0^\infty e^{-\lambda y} \cdot \lambda y^{n-1} \cdot \lambda^1\, dy$$

$$\Gamma n = \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} \times \lambda^1 \times y^{n-1}\, dy$$

$$= \int_0^\infty e^{-\lambda y} \lambda^{n-\cancel{1}+\cancel{1}}\, y^{n-1}\, dy$$

$$= \int_0^\infty e^{-\lambda y} \lambda^{n} y^{n-1}\, dy$$

$$\Gamma n = \lambda^n \int_0^\infty e^{-\lambda y} y^{n-1}\, dy \rightarrow (2)$$

Now Similary

$$\Gamma m = \lambda^m \int_0^\infty e^{-\lambda x} x^{m-1}\, dx \rightarrow (3)$$

Multiplying both side by $\underline{e^{-\lambda} \lambda^{n-1}}$ and integrating with respect to λ between the limits 0 and ∞, we get.

$x+1\cancel{+}x$

λ

$$\Gamma m \int_0^\infty e^{-\lambda} \lambda^{n-1}\, d\lambda = \lambda^m \int_0^\infty \left[\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1} \cdot d\lambda\right] x^{m-1}\, dx$$

$$\Gamma m \int_0^\infty e^{-\lambda} \lambda^{n-1}\, d\lambda = \int_0^\infty \left[\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1}\, d\lambda\right] x^{m-1}\, dx \rightarrow (4)$$

But $\Gamma n = \int_0^\infty e^{-\lambda} \lambda^{n-1}\, d\lambda$

$$\Gamma m\, \Gamma n = \int_0^\infty \left[\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1}\, d\lambda\right] x^{m-1}\, dx \rightarrow (4)$$

$$\boxed{\int_0^\infty e^{-\lambda(1-x)} \lambda^{m+n-1}\, d\lambda = \frac{\Gamma m+n}{(1+x)^{m+n}}} \rightarrow (5)$$

$$\Gamma m\, \Gamma n = \int_0^\infty \frac{\Gamma m+n}{(1+x)^{m+n}} \cdot x^{m-1}\, dx$$

$$\Gamma m \Gamma n = \Gamma m+n \int_0^\infty \frac{1}{(1+x)^{m+n}} \cdot x^{m-1} dx$$

$$\Gamma m \Gamma n = \Gamma m+n \; \beta(m,n)$$

$$\frac{\Gamma m \; \Gamma n}{\Gamma m+n} = \beta(m,n)$$

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1) $\Gamma(n+1) = n\Gamma n$

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx} \longrightarrow (1)$$

Replacing n by $(n+1)$ in eqn (1)

$n = n+1$

$$\Gamma n+1 = \int_0^\infty e^{-x} x^{n+1 \cancel{-1}} dx \qquad x^{n-1} : x^{n+1-1}$$

$$= \int_0^\infty \underbrace{e^{-x} \; x^n}_{u} \; \underbrace{dx}_{v} \qquad uv = uv' + vu'$$

dif $x^n = nx^{n-1}$

$e^x = e^x$

$$= \left[x^n e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} \; nx^{n-1} dx$$

$$= \cancel{x^n} \; 0 + n \underbrace{\int_0^\infty e^{-x} x^{n-1} dx}_{n} \text{ eqn (1)}$$

$$= n\Gamma n$$

$$\therefore \Gamma n+1 = n\Gamma n \; //$$

Fundamental Vecurrence (formula) Relation satisfy the gamma function.

ii) $\Gamma n = 1$

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx} \longrightarrow (1)$$

Put $n = 1$ in eqn (1)

$$\Gamma n = \int_0^\infty e^{-x} x^{1-1} dx$$

$$= \int_0^\infty e^{-x} dx$$

diff $e^{-x} = -e^{-x}$

$$= \left[-e^{-x}\right]_0^\infty$$

$e^{\infty} = 0$

$e^{0} = 1$

$$= -e^{-\infty} - e^{-0}$$

$$= [0 - (-1)]$$

$$= 1$$

$\therefore \Gamma n = 1$

Relation between gamma function and factorial

$$\Gamma(n+1) = n!$$

$$\Gamma(n+1) = n\Gamma(n) \longrightarrow (1)$$

Replacing n by $(n-1)$ in eqn (1)

$$\Gamma(n-\cancel{1}+\cancel{1}) = (n-1)\Gamma(n-1)$$

$$\boxed{\Gamma n = (n-1)\Gamma(n-1)}$$

Add -1 in both side

$$\Gamma(n-1) = (n-1-1)\Gamma(n-1-1)$$

$$\boxed{\Gamma(n-1) = (n-2)\Gamma(n-2)}$$

From eqn (1)

$$\Gamma(n+1) = n\Gamma n$$

$$= n(n-1)\Gamma(n-1)$$

Add -1 in both side

$$= (n-1)(n-1-1)\Gamma(n-1-1)$$

$$= (n-1)(n-2)\Gamma(n-3)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots 3,2,1\,\Gamma 1$$

$$= n(n-1)(n-2)(n-3)-\ldots 3,2,1$$

$$\Gamma(n+1) = n!$$

Fundamental properties :

$\Gamma(n+1) = n\Gamma n$

$\Gamma 1 = 1$

$\Gamma(n+1) = n!$

i) Value of $\Gamma\frac{3}{2}$

$\Gamma(n+1) = n\Gamma n$

Let $n = \frac{1}{2}$ $\quad \Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2}$

$= \frac{1}{2}\sqrt{\pi}$

$\Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}$

ii) Value of $\Gamma\left(-\frac{3}{2}\right)$

Let $n = \frac{-3}{2}$

$\Gamma\frac{-3}{2} = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$

$= \frac{\Gamma\frac{-3+2}{2}}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}}$

$$= -2\sqrt{\pi} \times -\frac{2}{3}$$

$$= \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

1) Show that $\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\Gamma(m+n)}$ $m>0$.

$$\Gamma m = \int_0^\infty e^{-s} (s)^{m-1} ds$$

$$\Gamma n = \int_0^\infty e^{-t} (t)^{n-1} dt$$

$$\Gamma m \, \Gamma n = \int_0^\infty e^{-s} (s)^{m-1} ds \int_0^\infty e^{-t} (t)^{n-1} dt$$

Let $s = x^2$, $t = y^2$

$ds = 2x\,dx$ $\quad dt = 2y\,dy$

$$\Gamma m \, \Gamma n = \int_0^\infty e^{-x^2} (x^2)^{m-1} \cdot 2x\,dx \int_0^\infty e^{-y^2} (y^2)^{n-1} \cdot 2y\,dy$$

$$= 4\int_0^\infty e^{-(x^2+y^2)} x^{2m-2+1} dx \int_0^\infty y^{2n-2+1} dy$$

$$\Gamma m \, \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \cdot x^{2m-1} \int_0^\infty y^{2n-2+1} dy$$

$$\Gamma m \, \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \cdot x^{2m-1} dx \cdot y^{2n-1} dy \rightarrow (1)$$

Change into polar form (or) coordinates

$x = r\cos\theta \quad , dx = dr$

$y = r\sin\theta \quad , dy = r\,d\theta$

$dx \cdot dy = r\,dr\,d\theta$

$x^2 + y^2 = r^2$ limits $= 0$ to $\pi/2$

$$\Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}(r\cos^{2m-1}\theta)(r\sin^{2n-1}\theta) \cdot r\,dr\,d\theta$$

$$\Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r^{(2m-1)+(2n-1)+1}\, dr \cdot \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\infty} e^{-r^2}\, r^{2m+2n-1}\, dr \cdot 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \to (2)$$

Let

$r^2 = z \qquad dr = \frac{1}{2} z^{\frac{1}{2}-1} dz$

$r = z^{1/2}$

$= \frac{1}{2} z^{-1/2} dz$

$dr = \frac{dz}{2z^{1/2}}$

$$= 2\int_0^{\infty} e^{-z}(z)^{\frac{2m+2n-1}{2}} \frac{dz}{2z^{1/2}} \Rightarrow \int_0^{\infty} e^{-z} z^{\frac{2m}{2}+\frac{2n}{2}-\frac{1}{2}} \cdot \frac{dz}{z^{1/2}}$$

$$= \int_0^{\infty} e^{-z} \cdot z^{m+n-\frac{1}{2}-\frac{1}{2}} \cdot dz$$

$$\Gamma m\,\Gamma n = \int_0^{\infty} e^{-z} z^{(m+n)-1} dz$$

$$\Gamma m\,\Gamma n = \int_0^{\infty} e^{-z} z^{(m+n)-1} dz$$

w.k.T

$$\Gamma n = \int_0^{\infty} e^{-x}(x)^{n-1} dx \qquad \because n = m+n$$

$$\Gamma m \, \Gamma n = \Gamma m+n \longrightarrow (3)$$

Sub in eqn (2)

$$\Gamma m \, \Gamma n = \Gamma m+n \; 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta$$

$$\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\,\Gamma m+n}$$

i) Prove that $\int_0^1 \frac{35x^3}{32\sqrt{1-x}}\, dx = 1$

Let

$x = \sin^2\theta$ $\qquad \sqrt{1-x} = \cos\theta$.

$dx = 2\sin\theta \cdot \cos\theta$

$1-x = 1-\sin^2\theta$

$1-x = \cos^2\theta$

limit

$x = 0$

$\theta = 0$

$x = 1$

$\theta = \pi/2$.

$$= \int_0^{\pi/2} \frac{35(\sin^2\theta)^3}{\cancel{32}_{16}(\cancel{\cos\theta})} \cdot \cancel{2}\sin\theta \cdot \cancel{\cos\theta}\, d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta \, d\theta$$

Compare the relation

$$\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\,\Gamma m+n}$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta \, d\theta$$

$m = 1/2$

$2n-1 = 7 \Rightarrow 2n = 8$

$n = 4$

$= \frac{35}{16}\left(\frac{\Gamma\frac{1}{2}\,\Gamma 4}{2\,\Gamma\frac{1}{2}+4}\right)$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma 4 = 3!$

$= 1\cdot 2\cdot 3$

$\Gamma\frac{9}{2} = \frac{7}{2}\Gamma\frac{7}{2} \Rightarrow \Gamma\frac{5}{2}+1$

$\Gamma n+1 = n\,\Gamma n$

$\Gamma\frac{1}{2}+4 = \Gamma\frac{9}{2}$

$\Gamma\frac{9}{2} = \Gamma\frac{7}{2}+1 = \frac{7}{2}\Gamma\frac{7}{2}$

$= \frac{7}{2}\,\frac{5}{2}\,\Gamma\frac{5}{2}$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\frac{3}{2}$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2} \Rightarrow \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$

$= \frac{\cancel{35}\,7}{16}\cdot\frac{\cancel{\sqrt{\pi}}\cdot 3\cdot\cancel{2}\cdot 1}{\cancel{2}\cdot\frac{\cancel{7}}{2}\cdot\frac{5}{2}\cdot\frac{\cancel{3}}{2}\cdot\frac{1}{2}\cancel{\sqrt{\pi}}}$

$= \frac{1}{\cancel{16}}\cdot 16$

$\int_0^1 \frac{35x^3}{32\sqrt{1-x}}\,dx = 1$

2) Prove that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

$\sqrt{\tan\theta} = \sqrt{\frac{\sin\theta}{\cos\theta}}$

$\int_0^{\pi/2}\sin^{1/2}\cos^{-1/2}\,d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma m+n}$

$2m-1 = \frac{-1}{2}$

$2n-1 = \frac{1}{2}$

$2m = -\frac{1}{2}+1$

$2n = \frac{1}{2}+1$

$2m = \frac{1}{2}$

$2n = \frac{3}{2}$

$m = \frac{1}{4}$

$n = \frac{3}{4}$

$= \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma\frac{1}{4}+\frac{3}{4}} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma\frac{4}{4}} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma 1} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2}$

3. S.T $\int_0^1 \frac{dx}{\sqrt{1-x^m}} = \frac{\Gamma 1/m}{\sqrt{(1/2+1/m)}} \cdot \frac{\sqrt{\pi}}{m}$

Let

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$x^m = \sin^2\theta$$

$$1-x^m = 1-\sin^2\theta = \cos^2\theta$$

$$\sqrt{1-x^m} = \cos\theta$$

$$x^{m/2} = \sin\theta$$

$$x^m = \sin^2\theta$$

$$x = \sin^{2/m}\theta$$

$$dx = \frac{2}{m}\sin^{2/m-1}\theta \cdot \cos\theta\, d\theta$$

$$L.H.S = \int_0^{\pi/2} \frac{\frac{2}{m}\sin^{2/m-1}\theta \cos\theta\, d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \frac{2}{m}\sin^{2/m-1}\theta\, d\theta$$

Comparing with the relation

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \frac{\Gamma m\,\Gamma n}{2\Gamma m+n}$$

$$2m-1 = 0 \qquad 2n-1 = \frac{2}{m}-1$$

$$2m = 1 \qquad n = 1/m$$

$$m = 1/2$$

$$L.H.S = \frac{\Gamma 1/2\, \Gamma 1/m}{2\Gamma 1/m+1/2} = \frac{2}{m}\frac{\Gamma 1/m \cdot \sqrt{\pi}}{\cancel{2}\Gamma 1/m + 1/2}$$

$$L.H.S = \frac{\Gamma 1/m\, \sqrt{\pi}}{m\Gamma 1/m + 1/2}$$

$$L.H.S = R.H.S$$

4. Show that $\int_0^1 x^{m-1}(1-x^a)^n\, dx = \frac{\frac{1}{a}\,\underline{\lfloor n}\,\Gamma m/a}{\Gamma(\frac{m}{a}+n+1)}$

Let

$$x^a = \sin^2\theta \Rightarrow x = \sin^{2/a}\theta$$

$$1-x^a = 1-(\sin^{2/a}\theta)^n = (\cos^2\theta)^n$$

$$dx = \frac{2}{a}\sin^{\frac{2}{a}-1}\theta \cos\theta\, d\theta$$

$$x^{m-1} = (\sin^{2/a}\theta)^{m-1} \Rightarrow x^{m-1} = \sin^{2/a \times m-1}\theta$$

$$L.H.S = \int_0^{\pi/2} \sin^{2/a(m-1)}\theta\, (\cos^2\theta)^n \cdot \frac{2}{a}\sin^{2/a-1}\theta \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{2}{a}\sin^{\frac{2m}{a}}\theta \cos^{2n+1}\theta\, d\theta \longrightarrow (1)$$

Comparing with the relation

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \frac{\Gamma m\, \Gamma n}{2\,\Gamma m+n}$$

$$2m-1 = 2n+1 \qquad\qquad 2n-1 = \frac{2m}{a} - 1$$

$$2m = 2n+1+1 \qquad\qquad 2n = \frac{2m}{a} - 1 + 1$$

$$2m = 2n+2$$

$$2m = 2(n+1) \qquad\qquad n = \frac{m}{a}$$

$$m = n+1$$

$$L.H.S = \frac{\Gamma n+1\; \Gamma m/a}{2\,\Gamma (n+1) + m/a} \qquad \because \Gamma n+1 = n! = \lfloor n$$

From eqn (1)

$$L.H.S = \frac{2}{a}\,\frac{\lfloor n\; \Gamma m/a}{2\;\Gamma (n+1) + m/a} = \frac{\frac{1}{a}\lfloor n\; \Gamma m/a}{m/a + (n+1)} = R.H.S$$

$$L.H.S = R.H.S$$

5. S.T $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\Gamma 1/n\, \sqrt{\pi}}{\sqrt{(1/2 + 1/n)}\cdot n}$

Let

$$x^n = \sin^2\theta \Rightarrow x = \sin^{2/n}\theta$$

$$1-x^n = 1-\sin^2\theta = \cos^2\theta$$

$$dx = \frac{2}{n}\sin^{2/n-1}\theta \cos d\theta$$

$$\sqrt{1-x^n} = \cos\theta$$

If $x = 0,\ \theta = 0$

$x = 1,\ \theta = \pi/2$

$$L.H.S = \int_0^{\pi/2} \frac{\frac{2}{n} \sin^{2/n - 1}\theta \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \int_0^{\pi/2} \frac{2}{n} \sin^{2/n-1}\theta \cancel{\cos\theta}\, d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{2/n-1}\theta\, d\theta$$

Comparing with the relation

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \frac{\Gamma m\, \Gamma n}{2\Gamma m+n}$$

$$2n-1 = \frac{2}{n} - 1 \qquad 2m-1=0$$

$$2n = \frac{2}{n} \qquad m = \frac{1}{2}$$

$$n = 1/n$$

$$= \frac{\cancel{2}}{n} \frac{\Gamma 1/2\, \Gamma 1/n}{\cancel{2}\,\Gamma 1/2 + 1/n}$$

$$= \frac{\Gamma 1/n \sqrt{\pi}}{\left(\frac{1}{2} + \frac{1}{n}\right) n} = R.H.S$$

$$L.H.S = R.H.S$$

6. S.T $\int_0^1 (1-x^n)^{1/n} dx$

$$x^n = \sin^2\theta$$

$$1-x^n = 1 - \sin^2\theta$$

$$(1-x^n)^{1/n} = (\cos^2\theta)^{1/n}$$

$$x = \sin^{2/n}\theta, \quad dx = \frac{2}{n}\sin^{2/n - 1}\theta \cdot \cos\theta\, d\theta$$

$$= \int_0^1 (\cos^2\theta)^{1/n} \cdot \frac{2}{n} \sin^{2/n-1}\theta \cdot \cos\theta\, d\theta$$

$$L.H.S = -\frac{2}{n}\int_0^1 \cos^{2/n+1}\theta \cdot \sin^{2/n-1}\, d\theta$$

Comparing with the relation

$$\int_0^{\pi/2} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \cdot d\theta = \frac{\Gamma m \, \Gamma n}{2\Gamma m+n}$$

$2m-1 = \frac{2}{n}+1$ | $2n-1 = \frac{2}{n}-1$

$2m = \frac{2}{n}+1+1$ | $2n = \frac{2}{n}$

$2m = \frac{2}{n}+2 \quad \frac{2}{2n}+\frac{2}{2} = \frac{1}{n}+1$ | $n = 1/n$

$m = \frac{1}{n}+1$

$$= \frac{\Gamma 1/n+1 \; \Gamma 1/n}{2\Gamma(1/n+1+1/n)}$$

$\Gamma \frac{2}{n}+1 = \frac{2}{n}\Gamma 2/n$

$\Gamma n+1 = n\Gamma n$

$$= \frac{\frac{1}{n}\Gamma 1/n \, \Gamma 1/n}{\cancel{\frac{2}{n}} \cdot \cancel{2}\Gamma \frac{2}{n}+1} = \frac{1}{\cancel{n}} \frac{\frac{1}{n}(\Gamma 1/n)^2}{\frac{2}{\cancel{n}}\Gamma 2/n}$$

$$L.H.S = \frac{\frac{1}{n}(\Gamma 1/n)^2}{2\Gamma 2/n} = R.H.S$$

7. S.T $\int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

$$I_1 = \int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}}$$

$$I_2 = \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$$

Let

$x^2 = \sin\theta$

$x = \sin^{1/2}\theta$

$1-x^4 = \sin^2\theta$

$1-x^4 = \cos^2\theta$

$(1-x^4)^{1/2} = (\cos^2\theta)^{1/2}$

$dx = \frac{1}{2}\sin^{\frac{1}{2}-1}\theta \cdot \cos\theta \, d\theta$

$= \frac{1}{2}\sin^{-1/2}\theta \cos\theta \, d\theta$

$dx = \frac{1}{2}\frac{\cos\theta}{\sin^{1/2}\theta} d\theta$

$\boxed{(1-x^4)^{1/2} = \cos\theta}$

Limits

$x=0 \Rightarrow \theta = 0$

$x=1 \Rightarrow \theta = \pi/2$

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta \frac{\cos\theta}{\sin^{1/2}\theta}}{\cos\theta} d\theta$$

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \, d\theta$$

$$I_2 = \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$$

$$\Gamma\tfrac{1}{4}+1 = n\Gamma n = \tfrac{1}{4}\Gamma\tfrac{1}{4}$$

$$= \frac{\frac{1}{2}\cdot\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{2\,\Gamma\frac{1}{2}+\frac{3}{4}} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{4\,\Gamma\frac{2}{4}+\frac{3}{4}} = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{4\,\Gamma\frac{5}{4}} = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{4\,\Gamma\frac{1}{4}+1}$$

$$I_1 = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{4\,\frac{1}{4}\,\Gamma\frac{1}{4}}$$

$$\boxed{I_1 = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{\Gamma\frac{1}{4}}}$$

$$I_2 = \int_0^1 \frac{dx}{(1-x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$$

$$x = \tan^{1/2}\phi$$

$$x^2 = \tan\phi$$

$$2x\,dx = \sec^2\phi\,d\phi$$

$$dx = \frac{\sec^2\phi}{2\tan^{1/2}\phi}\,d\phi$$

$$x^4 = \tan^2\phi$$

$$1+x^4 = 1+\tan^2\phi$$

$$= \sec^2\phi$$

$$(1+x^4)^{1/2} = \sec\phi$$

limits

$x=0,\ \phi=0$

$x=1,\ \phi=\pi/4$

$$I_2 = \int_0^{\pi/4} \frac{\frac{1}{2}\,\frac{\sec^2\phi}{2\tan^{1/2}\phi}\,d\phi}{\sec\phi} = \frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\phi}{2\tan^{1/2}\phi}\cdot\frac{1}{\sec\phi}\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{1}{\cos^2\phi}\cdot\frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{1}{\cos^{1/2}\phi\,\sin^{1/2}\phi}\,d\phi$$

multiple and limits in $\sqrt{2}$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sqrt{2}}{\sqrt{2\cos\phi\sin\phi}}\,d\phi = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\phi}{\sqrt{\sin 2\phi}}$$

Let

$$2\phi = \phi$$

$$2\, d\phi = d\phi$$

$$d\phi = \frac{d\phi}{2}$$

$$I_2 = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{d\phi}{2\sqrt{\sin\phi}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{\sin\phi}}$$

$$I_2 = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi \, d\phi$$

$$2m - 1 = 0 \qquad 2n - 1 = -\frac{1}{2}$$

$$m = 1/2 \qquad 2n = -\frac{1}{2} + 1$$

$$2n = \frac{1}{2} \Rightarrow n = \frac{1}{4} \qquad \frac{-1+2}{2}$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma 1/2 \; \Gamma 1/4}{2\,\Gamma\left(\frac{1}{2} + \frac{1}{4}\right)}$$

$$\frac{1\times 2}{2\times 2} + \frac{1}{4} \Rightarrow \frac{2+1}{4} = \frac{3}{4}$$

$$\boxed{I_2 = \frac{1}{4\sqrt{2}} \frac{\Gamma 1/2 \; \Gamma 1/4}{\Gamma 3/4}}$$

$$I_1 \times I_2 = \frac{\Gamma 1/2 \; \cancel{\Gamma 3/4}}{\cancel{\Gamma 1/4}} \times \frac{1}{4\sqrt{2}} \frac{\Gamma 1/2 \; \cancel{\Gamma 1/4}}{\cancel{\Gamma 3/4}}$$

$$= \frac{\Gamma 1/2 \; \Gamma 1/2}{4\sqrt{2}}$$

$$= \frac{\sqrt{\pi}\,\sqrt{\pi}}{4\sqrt{2}}$$

$$L.H.S = \frac{\pi}{4\sqrt{2}} = R.H.S$$

1) Relation b/w β and γ function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m,n >$$

Let

$$x = \cos^2\theta$$

$$dx = -2\cos\theta \cdot \sin\theta\, d\theta$$

$$1-x = 1-\cos^2\theta = \sin^2\theta$$

Formula $1-\cos^2\theta = \sin$

$$\beta(m,n) = -\int_{\pi/2}^{0} (\cos^2\theta)^{m-1}(\sin^2\theta)^{n-1}\, 2\cos\theta\sin\theta\, d\theta$$

limt chng
(-) sng chn

$$= 2\int_0^{\pi/2} \cos^{2m-2}\theta\, \sin^{2n-2}\theta\, \cos\theta\, \sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta$$

But

$$\int_0^{\pi/2} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta = \frac{\Gamma m\, \Gamma n}{2\,\Gamma m+n}$$

$$= \frac{2\,\Gamma m\, \Gamma n}{2\,\Gamma m+n}$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Corollary

$$\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\,\Gamma m+n}$$

Let

$2m-1 = p$ $\qquad$ $2n-1 = q$

$m = \frac{p+1}{2}$ $\qquad$ $n = \frac{q+1}{2}$

L.H.S

$$\frac{\Gamma \frac{p+1}{2} \, \Gamma \frac{q+1}{2}}{2\,\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)} = \frac{\Gamma \frac{p+1}{2} \, \Gamma \frac{q+1}{2}}{2\,\Gamma \frac{p+q+2}{2}}$$

Case (i) : $\int_0^{\pi/2} \cos^p\theta \, \sin^q\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\,\Gamma m+n}$

$p = 0, \; q = n$

$2m-1 = 0$ $\qquad$ $2n-1 = n$

$m = \frac{1}{2}$ $\qquad$ $2n = n+1$

$n = \frac{n+1}{2}$

$$\int_0^{\pi/2} \sin^q\theta \, d\theta = \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{\Gamma \frac{1}{2} \, \Gamma \frac{n+1}{2}}{2\,\Gamma \frac{n+2}{2}}$$

Case (ii) :

$P = n \quad , \quad q = 0$

$$\int_0^{\pi/2} \cos^P \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{\Gamma\frac{n+1}{2}\ \Gamma\frac{1}{2}}{2\ \Gamma\frac{n+2}{2}}$$

Case (iii) :

$P = 0 \ , \ q = 0$

$$\int_0^{\pi/2} d\theta = \frac{\Gamma\frac{1}{2}\ \Gamma\frac{1}{2}}{2\ \Gamma\frac{2}{2}} = \frac{\sqrt{\pi}\ \sqrt{\pi}}{2} = \frac{\pi}{2}$$

1) Prove that $\displaystyle\int_0^{\infty} \frac{x^3\, dx}{(1+x)^5} = \frac{1}{4}$

We know

$$\beta(m,n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy.$$

Comparing

$m - 1 = 3 \qquad m + n = 5$

$m = 4 \qquad 4 + n = 5 \qquad \Gamma 4 = 3!$

$n = 1 \qquad \Gamma 1 = 1!$

$$\beta(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma m+n} = \frac{\Gamma 4\ \Gamma 1}{\Gamma 5} \qquad \Gamma 5 = 4!$$

$$= \frac{3 \times 2 \times 1 \times 1}{4 \times 3 \times 2 \times 1} = \frac{6}{24} = \frac{1}{4} = R.H.S$$

$\therefore L.H.S = R.H.S$

2) $\displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}\;\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+2}{2}\right)}$

Let

$x^2 = \sin^2\theta$ $\qquad$ $x = \sin\theta$

$1-x^2 = 1-\sin^2\theta$ $\qquad$ $x^n = \sin^n\theta$

$\sqrt{1-x^2} = \cos\theta$ $\qquad$ $dx = \cos\theta\, d\theta$

$2m-1 = 0$ $\qquad$ $2n-1 = \frac{1}{2}$

$2m = 1$ $\qquad$ $2n = \frac{1}{2}+1$

$m = 1/2$ $\qquad$ $2n = \frac{3}{2} \Rightarrow n = \frac{3}{4}$

$$= \frac{1}{2}\,\frac{\Gamma(1/2)\,\Gamma(3/4)}{2\,\Gamma(1/2+3/4)} = \frac{1}{4}\,\frac{\Gamma(1/2)\,\Gamma(3/4)}{\Gamma(5/4)}$$

$$= \frac{1}{4}\,\frac{\Gamma(1/2)\,\Gamma(3/4)}{\frac{1}{4}\,\Gamma(1/4)} = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)}$$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)}$$

//

3) Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 0$

Let

$$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx$$

We know that

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^\infty \frac{x^{a-1}}{(1+x)^{9+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= \beta(9-15) - \beta(15-9)$$

$$= 0$$

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$$

$$\beta(2,4) = \frac{\Gamma 2\, \Gamma 4}{\Gamma 2+4} = \frac{\Gamma 2\, \Gamma 4}{\Gamma 6}$$

$\Gamma 4 = 3!$

$\Gamma 2 = 1!$

$\Gamma 6 = 5!$

$$= \frac{1! \times \cancel{3} \times \cancel{2} \times 1}{5 \times 4 \times \cancel{3} \times \cancel{2} \times 1}$$

$$= \frac{1}{20}$$