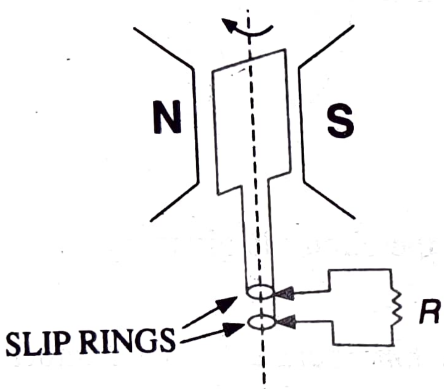
13

Alternating Current

13.1 EMF Induced in a Coil Rotating in a Magnetic Field

Consider a rectangular coil of N turns and of length a and width b rotating with uniform angular velocity ω about its axis in a uniform magnetic field **B** (Fig. 13.1).



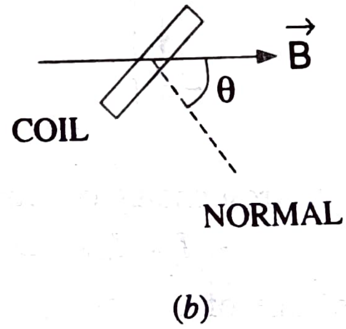


Fig. 13.1

The axis of rotation is at right angles to the field. As the coil rotates, the magnetic flux passing through it changes. Hence an emf is induced in the coil.

Suppose we start timing from the instant when the plane of the coil is at right angles to the field **B**, *i.e.*, when the angle between the normal to the plane of the coil and the direction of the field is zero. Then, at an instant t, the normal to the plane of the coil will make angle $\theta\ (=\omega t)$ with the direction of **B**.

The magnetic flux linked with N turns of the coil is

$$\phi = NBA\cos\theta = NBA\cos\omega t.$$

where $A\ (=ab)$ is the area of the coil.

The instantaneous induced emf,

$$E = -\frac{d\phi}{dt} = -\frac{d}{dt}(NBA\cos\omega t) = NBA\,\omega\sin\omega t = E_0\sin\omega t$$

Here, $E_0 = NBA\omega$, called the *peak value* of the e.m.f.

Now, $\omega = 2\pi\nu$ where ν = frequency of alternating voltage.

Thus when $\omega t = 0$, $\sin \omega t = 0$ and $E = 0$

$\omega t = \pi/2$, $\sin \omega t = 1$ and $E = E_0$

$\omega t = \pi$, $\sin \omega t = 0$ and $E = 0$

$\omega t = 3\pi/2$, $\sin \omega t = -1$ and $E = -E_0$

and $\omega t = 2\pi$, $\sin \omega t = 0$ and $E = 0$ again.

A graph of E against ωt is a sine curve (Fig. 13.2). Such an e.m.f. is called an 'alternating e.m.f.' The resulting current in the coil, if the coil is part of a *closed circuit*, is the 'alternating current.'

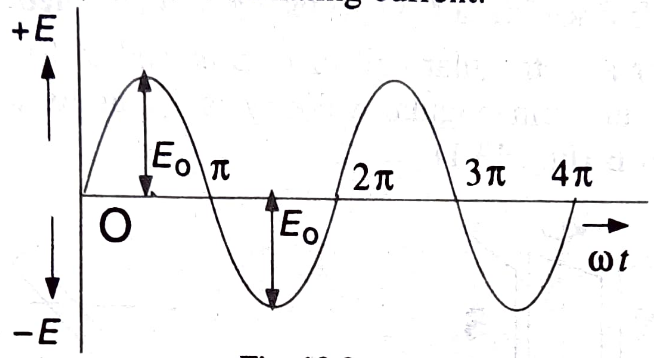


Fig. 13.2.

The corresponding current I through the circuit is given by

$$I = I_0 \sin \omega t.$$

The time of one cycle is known as *time period T*, the number of cycles per second the *frequency* ν (= $1/T$), the peak value of current or voltage the *amplitude*,

Peak Value of Alternating Current or emf. *The maximum value of alternating current or emf in the positive or negative direction is called peak value of alternating current or emf. It is denoted by* I_0 *or* E_0.

Mean Value of Alternating Current. *Mean value of alternating current is defined as its average over half a cycle.*

$$I_{mean} = \frac{\int_0^{T/2} I\,dt}{T/2} = \frac{\int_0^{\pi/\omega} I_0 \sin \omega t\,dt}{\pi/\omega}$$

$$= \frac{I_0\omega}{\pi}\left[\frac{-\cos \omega t}{\omega}\right]_0^{\pi/\omega}$$

$$= -\frac{I_0}{\pi}[\cos \pi - \cos 0]$$

$$= \frac{2I_0}{\pi} = 0.637\, I_0$$

Similarly, $E_{mean} = 0.637\, E_0$

Root mean square value of an alternating current. *It is defined as* *the square root of the average of I^2 during a complete cycle.*

$$\overline{I^2} = \frac{\int_0^{2\pi/\omega} I^2\, dt}{2\pi/\omega} = \frac{\int_0^{2\pi/\omega} I_0^2 \sin^2 \omega t\, dt}{2\pi/\omega}$$

$$= \frac{I_0^2 \omega}{2\pi} \int_0^{2\pi/\omega} \frac{1}{2}(1 - \cos 2\omega t)\, dt$$

$$= \frac{I_0^2 \omega}{4\pi} \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^{2\pi/\omega}$$

$$= \frac{I_0^2 \omega}{4\pi} \left[\frac{2\pi}{\omega} \right] = \frac{I_0^2}{2}$$

$$I_{rms} = \sqrt{\overline{I^2}} = \frac{I_0}{\sqrt{2}} = 0.707\, I_0$$

Similarly, $E_{rms} = \frac{E_0}{\sqrt{2}} = 0.707\, E_0$

Form factor. The form factor gives an indication of the wave shape of the alternating voltage or current. *It is defined as the ratio of the virtual or r.m.s. value to the average value of alternating current or voltage.* Thus in the case of a sinusoidal current (or voltage), form factor is,

$$\text{Form factor} = \frac{I_{r.m.s.}}{I_{mean}} = \frac{E_{r.m.s.}}{E_{mean}} = \frac{0.707\, E_0}{0.637\, E_0} = 1.11$$

Effective value or virtual value of an A.C. *The rms value of an alternating current can also be defined as that direct current which produces the same rate of heating in a given resistance. Therefore, the r.m.s. value of alternating current is also called as the 'effective' or the 'virtual' value of the current.*

$$I_{virtual} = \frac{I_0}{\sqrt{2}} = I_{rms}$$

Suppose an alternating current of instantaneous value $I = I_0 \sin \omega t$ is flowing through a circuit of resistance R.

$$\left.\begin{array}{l}\text{Total quantity of heat produced}\\ \text{over the complete cycle}\end{array}\right\} = H = \int_0^T I^2 R\, dt \qquad \text{... (1)}$$

Let I_v stand for the root mean square or virtual value of the current.

Then the heat produced in time T is given by

$$H = I_v^2 RT \qquad \text{... (2)}$$

Comparing Eqs. (1) and (2),

$$I_v^2 RT = \int_0^T I^2 R\, dt \quad \text{or} \quad I_v^2 T = \int_0^T I^2\, dt$$

or
$$I_v^2 T = \int_0^T I_0^2 \sin^2 \omega t\, dt = \frac{I_0^2}{2}\int_0^T 2\sin^2 \omega t\, dt$$

$$= \frac{I_0^2}{2}\int_0^T (1 - \cos 2\,\omega t)\, dt = \frac{I_0^2}{2}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{I_0^2 . T}{2}$$

or
$$I_v^2 = \frac{I_0^2}{2}$$

or
$$I_v = \frac{I_0}{\sqrt{2}} = 0.707\ I_0$$

Similarly, *the r.m.s. value of an alternating voltage can be defined as that direct voltage which produces the same rate of heating in a given resistance. The r.m.s. value of alternating voltage is also called as the 'effective' or the 'virtual' value of the voltage.*

$$E_{virtual} = \frac{E_0}{\sqrt{2}} = E_{rms.}$$

Impedance. *In any circuit the ratio of the effective voltage to the effective current is defined as the impedance Z of the circuit.*

13.2. AC Circuit Containing Resistance, Inductance and Capacitance in series (Series Resonance Circuit)

Let an alternating emf $E = E_0 \sin \omega t$ be applied to a circuit containing a resistance R, inductance L and capacitance C in series (Fig. 13.3). Let at any instant, I be the current in the circuit and Q be the charge on the capacitor.

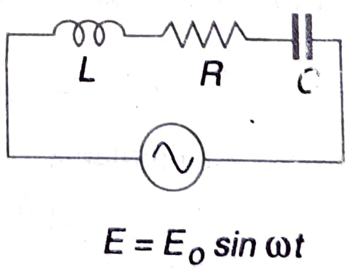


Fig. 13.3

The potential drop across the resistance = RI

The E.M.F. induced in the inductance = $L\dfrac{dI}{dt}$.

The potential across the plates of the capacitor = Q/C.

$$\therefore \qquad L\frac{dI}{dt} + RI + \frac{Q}{C} = E_0 \sin \omega t.$$

Differentiating with respect to t,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dQ}{dt} = E_0\omega\cos\omega t. \quad \dots (1)$$

Let the trial solution be of the form

$$I = I_0 \sin(\omega t - \phi), \quad \dots (2)$$

where I_0 and ϕ are constants to be determined.

$$\frac{dI}{dt} = I_0\omega\cos(\omega t - \phi)$$

and
$$\frac{d^2I}{dt^2} = -I_0\omega^2\sin(\omega t - \phi)$$

Substituting these values of I, $\frac{dI}{dt}$ and $\frac{d^2I}{dt^2}$, in Eq. (1), we get

$$-LI_0\omega^2\sin(\omega t - \phi) + RI_0\omega\cos(\omega t - \phi) + \frac{I_0}{C}\sin(\omega t - \phi) = E_0\,\omega\cos\omega t$$

or
$$\left(-L\omega^2 + \frac{1}{C}\right)I_0\sin(\omega t - \phi) + R\omega I_0\cos(\omega t - \phi)$$

$$= E_0\omega\,[\cos(\omega t - \phi) + \phi]$$
$$= E_0\omega\,[\cos(\omega t - \phi)\cos\phi - \sin(\omega t - \phi)\sin\phi]$$

Equating the coefficients of $\sin(\omega t - \phi)$ and $\cos(\omega t - \phi)$ on either side,

$$\left(-L\omega^2 + \frac{1}{C}\right)I_0 = -E_0\omega\sin\phi, \quad \dots (3)$$

and
$$R\omega I_0 = E_0\omega\cos\phi. \quad \dots (4)$$

Dividing Eq. (3) by Eq. (4), we get

$$\tan\phi = -\frac{\left(-L\omega^2 + \frac{1}{C}\right)}{R\omega} = \frac{\omega L - \frac{1}{C\omega}}{R} \quad \dots (5)$$

Squaring and adding Eqs. (3) and (4), we get

$$I_0^2\left[\left(-L\omega^2 + \frac{1}{C}\right)^2 + R^2\omega^2\right] = E_0^2\omega^2$$

or
$$I_0^2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right] = E_0^2$$

or $$I_0 = \frac{E_0}{\sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}} \quad \text{... (6)}$$

Substituting the value of I_0 in Eq. (2), we get

$$I = \frac{E_0}{\sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}} \sin(\omega t - \phi), \quad \text{... (7)}$$

where $\phi = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}$.

Eq. (7) represents the current at any instant.

The quantity $\sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}$ is the *impedance Z* of the circuit.

$L\omega$ and $1/\omega C$ respectively represent inductive reactance X_L and capacitive reactance X_C. Thus $Z = \sqrt{[R^2 + (X_L - X_C)^2]}$.

The current lags in phase behind e.m.f. by an angle

$$\phi = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R} = \tan^{-1} \frac{X_L - X_C}{R}$$

The following three cases arise :

(*i*) When $X_L > X_C$, ϕ is positive so that the *current lags behind the applied emf.*

(*ii*) When $X_L < X_C$, ϕ is negative, so that the *current leads the applied emf.*

(*iii*) When $X_L = X_C$, $\phi = 0$, and the *current is in phase with the emf.*

Series Resonant Circuit

The value of current at any instant in a series *LCR* circuit is given by

$$I = \frac{E_0}{\sqrt{\left\{R^2 + \left(\omega L - \frac{1}{C\omega}\right)^2\right\}}} \sin(\omega t - \phi) \quad \text{... (1)}$$

where $\sqrt{\left\{R^2 + \left(\omega L - \frac{1}{C\omega}\right)^2\right\}} = Z$

is called the impedance of the circuit.

At a particular frequency, $\omega L = \frac{1}{\omega C}$ so that the impedance becomes minimum being given by $Z = R$. *This particular frequency ν_0 at which the impedance of the circuit becomes minimum and, therefore the current becomes maximum, is called the* ***resonant frequency*** *of the circuit.* Such a circuit which admits maximum current is called *series resonant circuit.*

Thus at ν_0, we have

$$\omega L = \frac{1}{\omega C} \text{ or } 2\pi\nu_0 L = \frac{1}{2\pi\nu_0 C}$$

or
$$\nu_0 = \frac{1}{2\pi\sqrt{LC}}$$

The maximum current in the circuit = $I_0 = E_0/R$. The variation of current with frequency of applied voltage is shown in Fig. 13.4. The sharpness of peak depends upon the resistance R of the circuit. For low resistance, the peak is sharp.

Acceptor Circuit. The series resonant circuit is often called an 'acceptor' circuit. By offering *minimum impedance* to currents at the resonant frequency, it is able to select or accept most readily the current of this *one* frequency from among those of many frequencies.

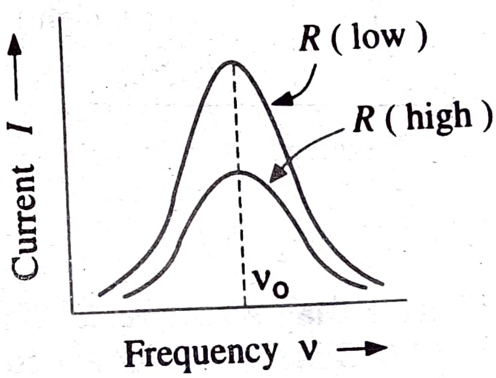


Fig. 13.4

In radio receivers, the resonant frequency of the circuit is tuned (by changing C) to the frequency of the signal desired to be detected.

Voltage Magnification. At resonance, the (peak) current through the circuit is

$$I_0 = \frac{E_0}{R}.$$

The (peak) voltage across the inductance is

$$V_L = X_L I_0 = \omega L \times \frac{E_0}{R} = \frac{\omega L}{R}(E_0) = Q E_0,$$

where Q is known as the *quality factor* of the circuit. Thus at resonance the voltage drop across L (which is also the voltage across C) is Q times the applied voltage. Hence the chief characteristic of the series resonant circuit is 'voltage magnification'. This voltage magnification does not increase the power in the circuit as the reactive component of the power is *wattless*.

The Q-factor

$$\text{Q-factor} = \frac{\text{Reactance of the coil at resonance}}{\text{Resistance of the circuit}} = \frac{L\omega_0}{R}$$

Q-factor determines the degree of selectivity of the circuit while tuning. This is because, for larger values of Q-factor the frequency response curve of the circuit is a steep narrow peak. For smaller values of Q-factor, the frequency response curve is quite flat (Fig. 13.5).

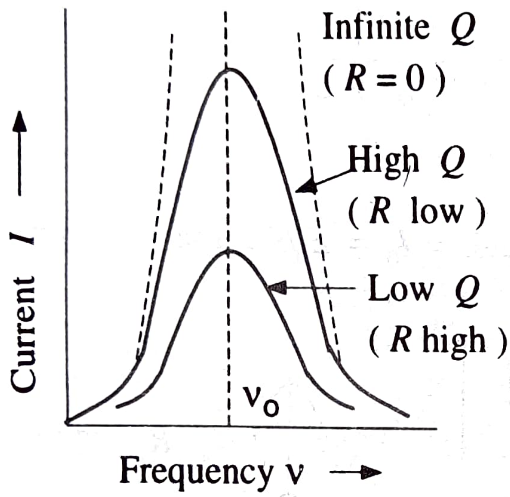


Fig. 13.5

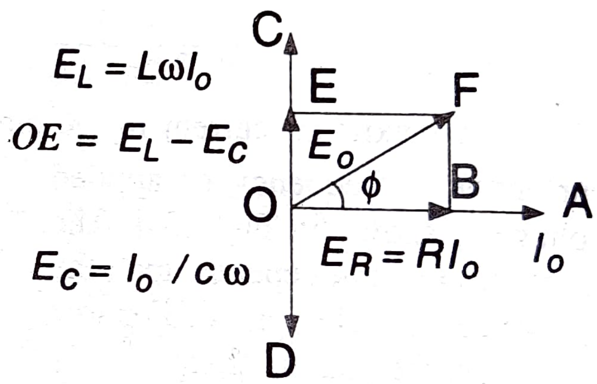


Fig. 13.6

Vector Diagram. A vector diagram of a series *LCR* circuit is shown in Fig. 13.6. Since $L-C-R$ are connected in series, the current through each is same. Let E_R, E_L and E_C be the potential drops across resistance, inductance and capacitance.

The vector $E_R = I_0 R$ is *in phase with* the current.

The vector $E_L = \omega L I_0$ is 90° *advance of* the current.

The vector $E_C = I_0/\omega C$, is 90° *behind* the current.

Let the vectors *OB*, *OC* and *OD* represent E_R, E_L and E_C. If $E_L > E_C$, the resultant of these two is $(E_L - E_C)$. This is represented by *OE*.

$$OE = OC - OD$$

$$= I_0\left(L\omega - \frac{1}{C\omega}\right); \text{ where } L\omega > \frac{1}{C\omega}$$

$$E_0 = [OB^2 + BF^2]^{\frac{1}{2}} = [OB^2 + OE^2]^{\frac{1}{2}}$$

$$= \left[I_0^2 R^2 + I_0^2 \left(L\omega - \frac{1}{C\omega} \right)^2 \right]^{\frac{1}{2}}$$

$$= I_0 \left[\left(L\omega - \frac{1}{C\omega} \right)^2 + R^2 \right]^{\frac{1}{2}}$$

$$I_0 = \frac{E_0}{\sqrt{\left[\left(L\omega - \frac{1}{C\omega} \right)^2 + R^2 \right]}}$$

The current lags behind the applied voltage by ϕ given as

$$\phi = \tan^{-1} \left(\frac{L\omega - 1/C\omega}{R} \right)$$

j operator method

Use of operator j in study of A.C. Circuits

The operator j is defined as a quantity which is numerically equal to $\sqrt{(-1)}$ and which represents the rotation of a vector through 90° in anti-clockwise direction. $-j$ represents the rotation of a vector through 90° in clockwise direction. The above facts about j are helpful in studying the A.C. circuits. We know that in A.C. circuits, E_L and E_C always lie at 90° in anticlockwise and clockwise direction respectively with respect to E_R (Fig. 13.7). Hence total emf of a circuit having L, C, R will be

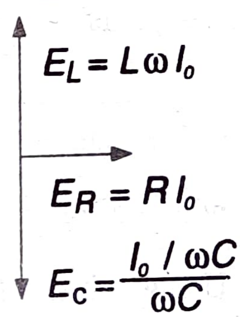


Fig. 13.7

$$E = E_R + jE_L - jE_C$$

Ordinarily a source of alternating e.m.f. E is denoted by $E_0 \sin \omega t$. This is actually the imaginary part of the complex form of alternating e.m.f.,

$$E = E_0 e^{j\omega t}$$

The instantaneous current in the A.C. circuit has been expressed as $I = I_0 \sin(\omega t - \phi)$. This expression is the imaginary part of the complex current given by

$$I = I_o e^{j(\omega t - \phi)}$$

Since the voltage across the inductor leads the current passing through it by 90°, the inductive reactance ωL can be written as $j\omega L$. $Z_L = j\omega L = jX_L$.

Since the voltage across the capacitor lags the current passing through it by 90°, the capacitive reactance $1/\omega C$ can be written as $-j/\omega C = 1/j\omega C$. $Z_c = \dfrac{1}{j\omega C} = -\dfrac{j}{\omega C} = -jX_C$.

A complex impedance can be written as the sum of a real term and imaginary term which are to be called resistance and complex reactance respectively,

$$Z = R + jX$$

where $X = X_L - X_C$ is the effective reactance in the circuit.

LCR Circuit (Series Resonance Circuit)

Consider a circuit containing an inductance L, a capacitance C and a resistance R joined in series. This series circuit is connected to an AC supply given by

$$E = E_0 e^{j\omega t} \quad \text{(Fig. 13.8)} \qquad \dots (1)$$

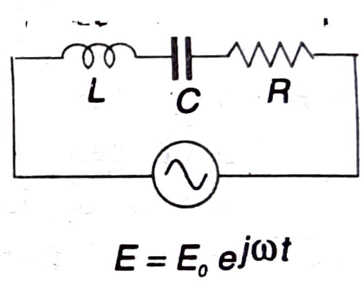


Fig. 13.8

The total complex impedance is

$$Z = Z_R + Z_L + Z_C$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$= \sqrt{R^2 + (\omega L - 1/\omega C)^2}\, e^{j\phi} \qquad \dots (2)$$

where $\tan\phi = \dfrac{(\omega L - 1/\omega C)}{R}$

Using Ohm's law in complex form, the 'complex' current in the circuit is

$$I = \frac{E}{Z} = \frac{E_0 e^{j\omega t}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\phi}}$$

$$\therefore \quad I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\, e^{j(\omega t - \phi)} \qquad \dots (3)$$

But $I_0 = \dfrac{E_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$

$\therefore \quad I = I_0 e^{j(\omega t - \phi)}$ ...(4)

The actual emf is the imaginary part of the equivalent complex emf. Hence the actual current in the circuit is obtained by taking the imaginary part of the above 'complex' current.

$$\therefore i = Im.(I) = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \sin(\omega t - \phi) \quad \text{... (5)}$$

The equivalent impedance of the series *LCR* circuit is

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The current 'lags' behind the voltage by an angle

$$\phi = \tan^{-1} \frac{(\omega L - 1/\omega C)}{R}$$

Example 1. *A circuit consists of a non-inductive resistance of 50 ohms, an inductance of 0.3 henry, and a resistance of 2 ohms and a capacitor of 40 micro-farad in series and is supplied with 200 volts at 50 Hz. Find the impedance, the current, lag or lead, and the power in the circuit.*

Sol. Total resistance = $R = 50 + 2 = 52\ \Omega$

$C = 40 \times 10^{-6}$ F, $L = 0.3$ H, $\omega = 2\pi\nu = 2\pi \times 50 = 100\pi$,

$E_{r.m.s.} = 200$ V.

$$\text{Impedance} = Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

$$= \sqrt{(52)^2 + \left(0.3 \times 100\pi - \frac{1}{40 \times 10^{-6} \times 100\pi}\right)^2}$$

$$= 53.97\ \Omega$$

$$\therefore \text{ Current } I_{r.m.s.} = \frac{E_{rms}}{Z} = \frac{200}{53.97} = 3.71\,A.$$

$$\therefore \quad I_{max} = 3.71 \times \sqrt{2}.\ A$$

Now, $$X_L = \omega L = (100\,\pi) \times 0.3 = 94.2\,\Omega$$

$$X_C = \frac{-1}{C\omega} = \frac{1}{(40 \times 10^{-6}) \times 100\,\pi} = 79.6\,\Omega$$

Since $X_L > X_C$, the current lags behind the applied e.m.f.

$$\phi = \tan^{-1} \frac{\omega L - 1/\omega C}{R} = \tan^{-1}\left(\frac{94.2 - 79.6}{52}\right) = \tan^{-1}(0.28)$$

$$= 15^{\circ}\ 39'$$

$$\text{Power factor } \cos\phi = \frac{R}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}$$

$$= \frac{52}{[(52)^2 + (14.6)^2]^{1/2}} = 0.964$$

Apparent power $= E_{rms} \times I_{rms} = 200 \times 3.71 = 742$ V.A

True power $=$ Apparent power $\times$ Power factor $= 742 \times 0.964$

$= 716$ watt

Example 2. *An alternating potential of 100 volt and 50 hertz is applied across a series circuit having an inductance of 5 henry, a resistance of 100 ohm and a variable capacitance. At what value of capacitance will the current in the circuit be in phase with the applied voltage ? Calculate the current in this condition. What will be the potential differences across the resistance, inductance and capacitance ?*

Sol. For resonance, $\omega L = \dfrac{1}{\omega C}$ or $C = \dfrac{1}{\omega^2 L}$

Here, $\omega = 2\pi\nu = 100\,\pi$, $L = 5$ H

$$\therefore C = \frac{1}{(100\pi)^2 \times 5} = 2 \times 10^{-6} \text{ farad} = 2\mu F$$

$$\left.\begin{array}{l}\text{The current}\\ \text{at resonance}\end{array}\right\} = I_{rms} = \frac{E_{rms}}{R} = \frac{100}{100} = 1.0\,A$$

P.D. across $R = I_{rms} \times R = 1.0 \times 100 =$ **100 volt (rms),**

P.D. across $L = I_{rms} \times \omega L = 1.0 \times (100\pi) \times 5 =$ **1570 volt (rms)**

$$\text{P.D. across } C = I_{rms} \times \frac{1}{C\omega} = 1.0 \times \frac{1}{(2 \times 10^{-6}) \times 100\,\pi}$$

$=$ **1570 volt (rms).**

The voltages across the inductor and capacitor are much greater than the applied voltage. But they differ in phase by 180^0. So their algebraic sum is zero.

13.3 Parallel Resonant Circuit

Here, capacitor C is connected in parallel to the series combination of resistance R and inductance L. The combination is connected across the AC source (Fig. 13.9). The applied voltage is sinusoidal, represented by

$$E = E_0 e^{j\omega t}$$

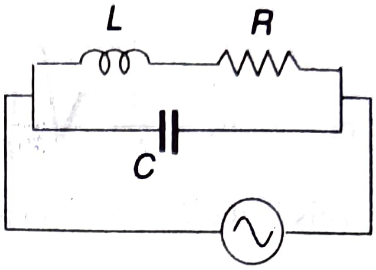


Fig. 13.9

Complex impedance of L - branch

$$Z_1 = R + jL\omega$$

Complex impedance of C - branch

$$Z_2 = \frac{1}{jC\omega}$$

Z_1 and Z_2 are in parallel.

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{R + j\omega L} + j\omega C$$

$$= \frac{R - j\omega L}{(R + j\omega L) \times (R - j\omega L)} + j\omega C$$

$$= \frac{R}{R^2 + (L\omega)^2} + j\left[C\omega - \frac{L\omega}{R^2 + (L\omega)^2}\right]$$

The current $I = E/Z = E \times \frac{1}{Z}$

$$\therefore \quad I = E\left[\frac{R}{R^2 + (L\omega)^2} + j\left(C\omega - \frac{L\omega}{R^2 + (L\omega)^2}\right)\right]$$

Let $A\cos\phi = \frac{R}{R^2 + (L\,\omega)^2}$; $A\sin\phi = C\omega - \frac{L\omega}{R^2 + (L\omega)^2}$

$$\therefore \quad I = E\,(A\cos\phi + j\,A\sin\phi) = EAe^{j\phi} = E_0 A\, e^{j(\omega t + \phi)}$$

where $$\phi = \tan^{-1}\frac{C\omega - \left(\dfrac{L\omega}{R^2 + (L\omega)^2}\right)}{\left(\dfrac{R}{R^2 + (L\omega)^2}\right)};$$

$$A^2 = \frac{R^2}{(R^2 + \omega^2 L^2)^2} + \left(C\omega - \frac{L\omega}{R^2 + \omega^2 L^2}\right)^2$$

The magnitude of the admittance

$$Y = \frac{1}{Z} = \frac{\sqrt{[R^2 + (\omega CR^2 + \omega^3 L^2 C - \omega L)^2]}}{R^2 + \omega^2 L^2}$$

The admittance will be minimum, when

$$\omega CR^2 + \omega^3 L^2 C - \omega L = 0,$$

or $$\omega = \omega_0 = \sqrt{\left[\frac{1}{LC} - \frac{R^2}{L^2}\right]}$$

or $$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is the resonant frequency of the circuit.

If R is very small so that $\frac{R^2}{L^2}$ is negligible compared to $\frac{1}{LC}$,

$$\nu_0 = \frac{1}{2\pi\sqrt{(LC)}}$$

At such a minimum admittance, *i.e.*, maximum impedance, the circuit current is minimum.

The graph between current and frequency is shown in Fig. 13.10.

Impedance at Resonance

At resonance, $Z = \frac{R^2 + (L\omega)^2}{R}$

But $R^2 + (L\omega)^2 = \frac{L}{C}$ at resonance

$\therefore$ $$Z = \frac{L}{RC}$$

Thus smaller the resistance R, larger is the impedance. If R is negligible, the impedance is infinite at resonance.

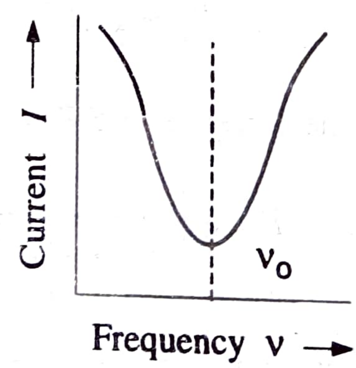


Fig. 13.10

Rejector Circuit. The parallel resonant circuit does not allow the current of the same frequency as the natural frequency of the circuit. Thus it can be used to suppress the current of this particular frequency out of currents of many other frequencies. Hence the circuit is known as a 'rejector' or 'filter' circuit.

Comparison between series and parallel resonant circuits

The behaviour of a parallel resonant circuit is strikingly different from that of a series resonant circuit. In both cases the impedance is resistive but, whereas parallel resonance implies maximum impedance, series resonance implies minimum impedance.

	Series resonant circuit	Parallel resonant circuit
1.	An acceptor circuit.	A rejector circuit.
2.	Resonant frequency $\nu_r = \frac{1}{2\pi\sqrt{LC}}$	Resonant frequency $\nu_r = \frac{1}{2\pi\sqrt{LC}}$
3.	At resonance the impedance is a minimum equal to the resistance in the circuit.	At resonance the impedance is maximum nearly equal to infinity.
4.	Selective.	Selective.
5.	Used in the tuning circuit to separate the wanted frequency from the incoming frequencies by offering low impedance at that frequency.	Used to present a maximum impedance to the wanted frequency, usually in the plate circuit of valves.

Example 1. *A coil of self inductance 2 milli-henry and resistance 15 ohm is connected in parallel with a capacitance of 0.001 μF. Find (a) the frequency at which the current from an a.c. source to this circuit is minimum, (b) the peak-value of this make-up current if the peak value of supply voltage is 2 volt.*

Sol. (*a*) Current is minimum at resonant frequency.

$$\nu_0 = \frac{1}{2\pi}\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)^{1/2}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(2\times 10^{-3})(0.001\times 10^{-6})} - \frac{(15)^2}{(2\times 10^{-3})^2}\right]^{1/2} = 112590\text{Hz}$$

Aliter. $\nu_0 = \frac{1}{2\pi}\frac{1}{\sqrt{(LC)}}\left(\text{as } \frac{R^2}{L^2} << \frac{1}{LC}\right)$

$$= \frac{1}{2\times 3.14\times\sqrt{(2\times 10^{-3}\times 0.001\times 10^{-6})}} = 112596 \text{ Hz}$$

(*b*) Peak-value of make-up current at resonance

$$= \frac{\text{peak value of applied e.m.f.}}{\text{impedanceat resonance}} = \frac{E_0}{L/CR} = \frac{E_0 CR}{L}$$

$$= \frac{2\times(0.001\times 10^{-6})\times 15}{2\times 10^{-3}} = 15\times 10^{-6}\ A$$

Power Series

13.4. Power in ac circuit containing resistance, inductance and capacitance

Consider an ac circuit containing resistance, inductance and capacitance. E and I vary continuously with time. Therefore power is calculated at any instant and then its mean is calculated over a complete cycle.

The instantaneous values of the voltage and current are given by

$$E = E_0 \sin \omega t,$$

$$I = I_0 \sin (\omega t - \phi).$$

where ϕ is the phase difference between current and voltage.

Hence power at any instant is

$$E \times I = E_0 I_0 \sin \omega t \sin (\omega t - \phi)$$

$$= \frac{1}{2} E_0 I_0 [\cos \phi - \cos (2\omega t - \phi)]. \quad \dots (1)$$

Average power consumed over one complete cycle is

$$P = \frac{\int_0^T E\,I dt}{\int_0^T dt}$$

$$= \frac{\int_0^T \frac{1}{2} E_0 I_0 [\cos \phi - \cos (2\omega t - \phi)]\, dt}{T}$$

$$= \frac{1}{2} \frac{E_0 I_0}{T} \left[(\cos \phi)\, t - \frac{\sin (2\omega t - \phi)}{2\omega} \right]_0^T$$

$$= \frac{1}{2} \frac{E_0 I_0}{T} \left[(\cos \phi)\, T - 0 - \frac{\sin (2\omega T - \phi)}{2\omega} + \frac{\sin (-\phi)}{2\omega} \right]$$

Now $T = \frac{2\pi}{\omega}$ and $\sin (4\pi - \phi) = \sin (-\phi)$

$$P = \frac{1}{2} \frac{E_0 I_0 \omega}{2\pi} \left[(\cos \phi) \frac{2\pi}{\omega} - \frac{\sin (-\phi)}{2\omega} + \frac{\sin (-\phi)}{2\omega} \right]$$

$$= \frac{1}{2} E_0 I_0 \cos \phi$$

$$= \frac{E_0}{\sqrt{2}} \times \frac{I_0}{\sqrt{2}} \times \cos \phi$$

$$= E_{r.m.s.}\, I_{r.m.s.} \cdot \cos \phi \quad \dots (2)$$

average power = (virtual volts) × (virtual amperes) × cos ϕ

The term (*Virtual volts* × *Virtual amperes*) is called *apparent power* and cos ϕ is called the *power factor*. Thus

True power = *apparent power* × *power factor*

Evidently, the *power factor is the ratio of the true power to the apparent power.*

As cos ϕ is the factor by which the product of the r.m.s. values of the voltage and current must be multiplied to give the power dissipated, it is known as the 'power factor' of the circuit. For a circuit containing resistance, capacitance and inductance in series,

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

From Fig. 13.11, the expression for the power factor is

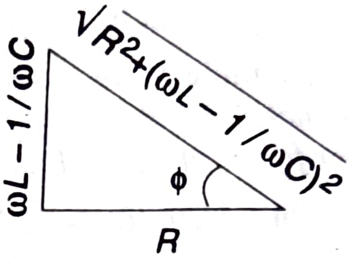


Fig. 13.11

$$\cos\phi = \frac{R}{\sqrt{\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}}}$$

Special cases. (1) In a *purely resistive circuit*, $\phi = 0$ or $\cos\phi = 1$.

∴ true power $= E_v \times I_v$.

(2) In a *purely inductive circuit*, current lags behind the applied emf by 90° so that $\phi = 90^\circ$ or $\cos\phi = 0$.

Thus true power consumed = 0

(3) In a *purely capacitive circuit*, current leads the applied voltage by 90° so that $\phi = -90^\circ$ or $\cos(-90^\circ) = \cos 90^\circ = 0$

∴ true power = 0

(4) In an ac circuit containing a resistance and inductance in series,

$$\textit{Power factor}, \cos\phi = \frac{R}{\sqrt{R^2 + (L\omega)^2}}$$

(5) In an ac circuit containing a capacitance C and a resistance R in series,

$$\cos\phi = \frac{R}{\sqrt{\left(\frac{1}{C^2\omega^2} + R^2\right)}}$$

13.5 Wattless current

The average power dissipated during a complete cycle is $E_v I_v \cdot \cos\phi$

The current in A.C. circuit is said to be wattless when the average power consumed in the circuit is zero.

If an ac circuit is purely inductive or purely capacitive with no ohmic resistance, phase angle $\phi = \pi/2$ so that $\cos\phi = 0$ or the power consumed is zero. The current in such a circuit does not perform any useful work and is rightly called the *wattless* or *idle current.* In this situation, the circuit does not consume any power, though it offers a resistance to the flow of alternating current in it. It is the principle of choke coil.

Example 1. *An alternating voltage of 10 volts at 100 Hz is applied to a choke of inductance 5 henry and of resistance 200 ohms. Find the power factor of the coil and the power absorbed.*

Sol. Here, $E_v = 10\,V$, $\nu = 100\,Hz$, $L = 5\,H$, and $R = 200\,\Omega$.

$$\text{Power factor, } \cos\phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \frac{200}{\sqrt{(200)^2 + (2\pi \times 100 \times 5)^2}}$$

$$= 0.062$$

$$\text{Power absorbed} = E_v \cdot I_v \cdot \cos\phi = E_v \cdot \frac{E_v}{Z} \cos\phi$$

$$= \frac{(E_v)^2}{\sqrt{R^2 + \omega^2 L^2}} \cos\phi = \frac{10 \times 10\,(0.062)}{\sqrt{(200)^2 + (2\pi \times 100 \times 5)^2}}$$

$$= 0.00189$$

Example 2. *A source of e.m.f. 50 V, r.m.s. is connected across an air cored coil. When the supply frequency is 50 Hz the power consumed is found to be 100 W, whereas when the frequency is increased to 100 Hz, the power becomes 50 W. Find the self-inductance and the resistance of the coil.*

Sol. Let the inductance and resistance required be L and R respectively.

$$I_v = \frac{E_v}{\sqrt{(R^2 + \omega^2 L^2)}}$$

When the supply frequency is 50 Hz, $\omega = 2\pi \times 50. = 100\pi$

$$I_v' = \frac{50}{\sqrt{R^2 + (100\pi)^2 L^2}}$$

The power consumed is dissipated only in the circuit resistance. It is,

$$I_v'^2 R = \frac{50^2}{[R^2 + (100\pi)^2 L^2]} R = 100W \qquad \text{.... (1)}$$

$\therefore$ current, $I_{rms} = \dfrac{E_{rms}}{Z} = \dfrac{200}{225.6} = 0.89$ A

Power component of current is

$$I_{rms} \cos \phi = I_{rms} \times \frac{R}{\sqrt{(R^2 + \omega^2 L^2)}} = 0.89 \times \frac{50}{225.6}$$

$$= 0.20 \text{ A}$$

The wattless component of the current is

$$I_{rms} \sin \phi = I_{rms} \times \frac{\omega L}{\sqrt{(R^2 + \omega^2 L^2)}} = 0.89 \times \frac{220}{225.6}$$

$$= 0.87$$

Example 4. *A coil has an inductance of 0.1 H and a resistance of 12 ohms. It is connected to a 220 V, 50 Hz mains. Determine the (1) reactance of the coil, (2) impedance of the coil, and (3) the reading of a wattmeter.*

Sol. Reactance of coil $= L\omega = 0.1 \times 2\pi \times 50\ \Omega = 31.43\ \Omega$

Z = Impedance of coil $= \sqrt{R^2 + L^2\omega^2}$

$$= \sqrt{(12)^2 + 4\pi^2 \times (0.1)^2 \times (50)^2} = 33.6\ \Omega$$

Reading of watt meter $= \bar{P} = \dfrac{E_{rms}^2}{Z} \cos \phi$

$$= \frac{(220)^2}{33.6} \times \left(\frac{12}{33.6}\right)$$

$$= 514.5 \text{ W}$$

Example 5. *There is no dissipation of power when an alternating emf is applied to a purely inductive circuit. Explain.*

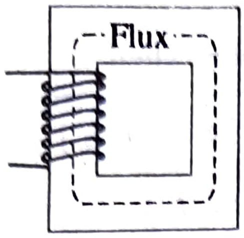
Sol. The power used by the source in one part of a cycle to create the magnetic field of L is delivered back to the source in the remaining part of the cycle. Thus the inductance acts like a storage of magnetic energy in the circuit. This storage gets filled up and emptied alternately, as the voltage source drives the circuit. No energy is wasted in this process. In the resistive part (R) of the coil, some losses occur by way of heat dissipation.

13.6. Choke Coil

A choke coil is an inductance coil which is used to control the current in an ac circuit.

Construction. A choke consists of a coil of several turns of insulated thick copper wire of low resistance but large inductance, wound over a laminated core (Fig. 13.13). The core is layered and is made up of thin sheets of stalloy to reduce hysteresis losses. The laminations are coated with shellac to insulate and bound together firmly so as to minimise loss of energy due to eddy currents.

Principle.



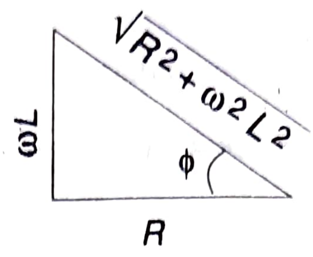


Fig. 13.13

Fig. 13.14

The average power dissipated in the choke coil is given by

$$P = \frac{1}{2} E_0 I_0 \cos \phi$$

If the resistance of the choke coil is R and the inductance of the choke coil is L, then the power factor $\cos \phi$ is given by

$$\cos \phi = \frac{R}{\sqrt{(R^2 + \omega^2 L^2)}} \qquad \text{(Fig. 13.14)}$$

The inductance L of the choke coil is quite large on account of its large number of turns and the high permeability of iron core, while its resistance R is very small. Hence $\cos \phi$ is nearly zero. Therefore, the power absorbed by the coil is extremely small. Thus *the choke coil reduces the strength of the current without appreciable wastage of energy.* The only waste of energy is due to the hysteresis loss in the iron core. The loss due to eddy currents is minimised by making the core *laminated.*

Preference of choke coil over an ohmic resistance for diminishing the current. The current in an A.C. circuit can also be diminished by using an ordinary ohmic resistance (rheostat) in the circuit. But such a method of controlling A.C. is not economical as much of the electrical energy (I^2Rt) supplied by the source is wasted as heat. Hence the choke coil is to be preferred over the ohmic resistance.

The energy used in establishing the magnetic field in the choke coil is restored when the magnetic field collapses. Hence to regulate ac, it is more economical to use a choke than a resistance.

Choking coils are very much used in electronic circuits, mercury lamps and sodium vapour lamps.

Example 1. *An electric lamp which runs at 100 volts D.C. and 10 amp. current is connected to 220 volts 50 Hz A.C. mains. Calculate the inductance of the choke in the circuit.*

Sol. Resistance of the lamp $R = \frac{V}{I} = \frac{100}{10} = 10\,\Omega$.

If the lamp is to be run from 220 volts, 50 Hz A.C. mains a choke (inductance) should be placed in series with the lamp in order to increase its effective resistance.

Let L be the inductance of the required choke. Then

$$\text{Impedance} = \sqrt{(R^2 + \omega^2 L^2)} = \sqrt{[(10)^2 + (2\pi \times 50)^2 L^2]}$$

$$= \sqrt{(100 + 10^4 \pi^2 L^2)}.$$

Now $$\text{current} = \frac{\text{voltage}}{\text{impedance}}$$

$$\therefore \quad 10 = \frac{220}{\sqrt{(100 + 10^4 \pi^2 L^2)}}$$

$$\therefore \quad L = 0.062\ H.$$

Example 2. *A 20 V, 5 W lamp is to be used on ac mains of 200 V, 50 Hz. Calculate the (i) capacitor, (ii) inductor, to be put in series to run the lamp. How much pure resistance should be included in place of the above devices so that the lamp can run on its rated voltage ? Which of the above arrangements will be more economical and why ?*

Sol. $$\left.\begin{array}{l}\text{Current required}\\ \text{by the lamp}\end{array}\right\} = I = \frac{\text{wattage}}{\text{voltage}} = \frac{5}{20} = 0.25\ A$$

$$\left.\begin{array}{l}\text{Resistanceof}\\ \text{the lamp}\end{array}\right\} = R = \frac{\text{voltage}}{\text{current}} = \frac{20}{0.25} = 80\ \Omega$$

(*i*) When a capacitor of value C farads is put in series with the lamp,

$$\text{the impedance of the circuit} = \sqrt{\left(R^2 + \frac{1}{\omega^2 C^2}\right)}$$

$$I = \frac{E}{\sqrt{(R^2 + 1/\omega^2 C^2)}}$$

or $$\frac{200}{\sqrt{\left(80^2 + \frac{1}{4\pi^2 \times 50^2 C^2}\right)}} = 0.25$$

$$\therefore\ C = 4.0 \times 10^{-6}\ \text{F} = 4.0\ \mu F.$$

(ii) When an inductor of value L henry is put in series with the lamp,

the impedance of the circuit = $\sqrt{(R^2 + \omega^2 L^2)}$

$$I = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}}$$

$$\frac{200}{\sqrt{(80^2 + 4\pi^2 \times 50^2 L^2)}} = 0.25$$

$\therefore$ $L = 2.53$ H

(iii) When a resistance of $r\ \Omega$ is put in series with the lamp,

$$I = \frac{E}{R + r}$$

$$\frac{200}{80 + r} = 0.25$$

$\therefore$ $r = 720\ \Omega$

(iv) The insertion of a resistance of 720 Ω to run the lamp at its rated value will cause a dissipation of power. However, the use of pure inductance or of pure capacitance consumes no power. Therefore, it will be more economical to use inductance or capacitance.

13.7. The Transformer

It is a device for converting a low alternating voltage at high current into a high alternating voltage at low current and vice-versa. It is an electrical device based on the principle of mutual induction between the coils.

Construction. A transformer consists of two coils, called the *primary P* and *secondary S*, which are insulated from each other and wound on a common soft-iron laminated core (Fig. 13.15).

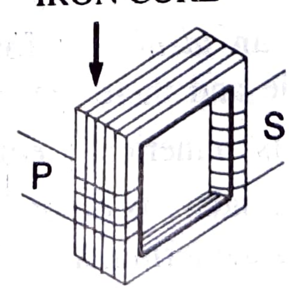


Fig. 13.15

The alternating voltage to be transformed is connected to the primary while the load is connected to the secondary. Transformers which convert low voltages into higher voltages are called *step-up* transformers. Transformers which convert high voltages into lower voltages are called *step-down* transformers.

In a step-up transformer, the primary coil consists of a few turns of thick insulated copper wire of large current carrying capacity and secondary consists of a very large number of turns of thin copper wire. In a step-down transformer, the primary consists of a large number of turns of thin copper wire and the secondary of a few turns of thick copper wire.

Now when an ac is applied to the primary coil, it sets up an alternating magnetic flux in the core which also gets linked with the secondary. This change in flux linked with the secondary coil induces an alternating emf in the secondary coil. Thus the energy supplied to the primary is transferred to the secondary through the changing magnetic flux in the core.

Theory

(i) Transformer on no load

Let N_p and N_S be the number of turns in the primary and secondary respectively of the transformer (Fig. 13.16). When an alternating emf is applied across the primary, a current flows in its winding. This develops a magnetic flux in the core. Here it is assumed that there are no losses and no leakage of flux. Let ϕ = flux linked with each turn of either coil. This magnetic flux is linked up with both the primary and the secondary.

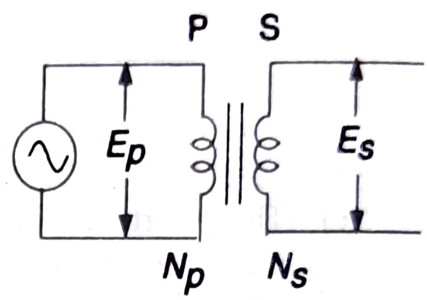


Fig. 13.16

By Faraday's law of electromagnetic induction, the emf induced in the primary is given by

$$\varepsilon_p = -\frac{d(N_p\phi)}{dt} = -N_p\frac{d\phi}{dt}$$

and the emf induced in the secondary,

$$\varepsilon_s = -\frac{d(N_s\phi)}{dt} = -N_s\frac{d\phi}{dt}$$

$$\therefore \quad \frac{\varepsilon_s}{\varepsilon_p} = \frac{N_s}{N_p}.$$

In an *ideal* transformer, the resistance of the primary circuit is negligible and there are no energy losses. So the induced emf ε_p in the primary is numerically equal to the applied voltage E_p across the primary. If the secondary is on open circuit, its resistance is infinite. So the voltage E_s across the terminals of the secondary is equal to the induced emf ε_s.

$$\frac{E_s}{E_p} = \frac{\varepsilon_s}{\varepsilon_p} = \frac{N_s}{N_p} = K, \qquad \text{... (1)}$$

where K is called the *turns ratio* or *transformation ratio* of the transformer.

$$K = \frac{\text{voltage obtained across secondary}}{\text{voltage applied across primary}} = \frac{\text{No. of turns in secondary}}{\text{No. of turns in primary}}$$

In a step-up transformer $\varepsilon_s > \varepsilon_p$ and hence $N_s > N_p$.

In a step-down transformer $\varepsilon_s < \varepsilon_p$ and hence $N_s < N_p$.

Let I_p and I_s be the currents in primary and secondary at any instant.

In this case power output is equal to power input.

power in the secondary = power in the primary

$$E_s \times I_s = E_p \times I_p$$

$$\frac{I_p}{I_s} = \frac{E_s}{E_p} = \frac{N_s}{N_p} = K. \quad \text{... (2)}$$

Thus, *when the voltage is stepped-up, the current is correspondingly reduced in the same ratio, and vice-versa.*

(ii) Transformer on load

If the primary circuit has an appreciable resistance R_p, the difference between the applied voltage E_p and the back e.m.f. ε_p must be equal to the potential drop $I_p \times R_p$ in the primary coil, *i.e.*,

$$E_p - \varepsilon_p = I_p \times R_p.$$

or

$$\varepsilon_p = E_p - I_p \times R_p \quad \text{... (3)}$$

Again, if the secondary circuit is closed having finite resistance (load) R_s, a part of the induced emf ε_s in the secondary overcomes the potential drop $i_s \times R_s$ (Fig. 13.17). Hence the available P.D. across the secondary is given by

$$E_s = \varepsilon_s - I_s \times R_s$$

or

$$\varepsilon_s = E_s + I_s \times R_s$$

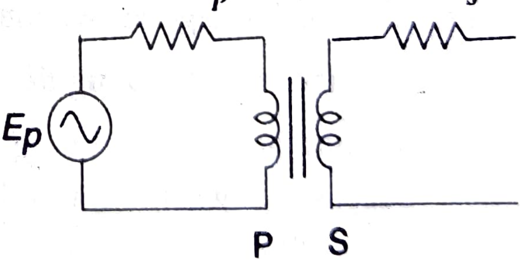


Fig. 13.17

Hence
$$\frac{\varepsilon_s}{\varepsilon_p} = \frac{E_s + I_s R_s}{E_p - I_p R_p} = K$$

or $E_s + I_s R_s = K(E_p - I_p R_p)$

$$\therefore \quad E_s = KE_p - I_s R_s - KI_p R_p$$

$$= KE_p - I_s R_s - K^2 I_s R_p \quad (\because I_p = K I_s)$$

$$= KE_p - I_s (R_s + K^2 R_p)$$

In this case E_s/E_p is not a constant but decreases as more current is drawn from the secondary circuit.

Energy Losses in a Transformer

Only in an ideal transformer the power output is equal to the input power. In actual transformers the output power is always less than the input power because of unavoidable energy losses. These losses are :

(1) *Copper loss.* There is loss of power due to Joule heating in the primary and secondary windings.

(2) *Iron loss.* This is due to the eddy currents being produced in the core of the transformer. This is minimised by using a laminated iron core.

(3) *Hysteresis loss.* During each cycle of A.C., the core is taken through a complete cycle of magnetisation. The energy expended in this process is finally converted into heat and is, therefore, wasted. This is minimised by using silicon-iron for preparing the core. The hysteresis loop for this material is very narrow.

(4) *Leakage of magnetic flux.* Due to leakage, all the magnetic flux produced in the core by the primary is not linked with the secondary. They may pass through air. The loss due to this cause is minimised by using a shell type of core.

Uses of Transformers

(1) The *step-up* and *step-down transformers* are used in a.c. electrical power distribution for the domestic and industrial purposes.

(2) The *audio-frequency transformers* are used in radio receivers, radio-telephony, radio-telegraphy and in televisions.

(3) The *radio frequency transformers* are used in radio-communications at frequencies of the order of mega-cycles.

(4) The *impedance transformers* are used for matching the impedance between two circuits in radio communication.

(5) The constant current and constant voltage transformers are designed to give constant output current and voltage respectively even when the input voltage varies considerably.

Use of transformers in long distance power transmission :

Advantage of high voltage in transmission :

Suppose we want to transmit a given power (VI) = 44,000 W from a generating station to a distant city. It can be transmitted (*i*) at a voltage of 220 V and a current of 200 A or (*ii*) at a voltage of 22000 V and a current of 2 A. Following are the losses which occur in the transmission :

(*a*) When the current is flowing through the line wires, the energy (I^2Rt) will be lost as heat. This would be greater in the first case.

(*b*) The voltage drop along the line wire is equal to (RI). Again this loss is greater in the first case.

(*c*) The line wires, which are to carry the *high* current, will have to be made thick. Such wires will be expensive.

Thus from the above example it is clear that from the point of view of both efficiency and economy, the power must be transmitted at high voltage and at low current.