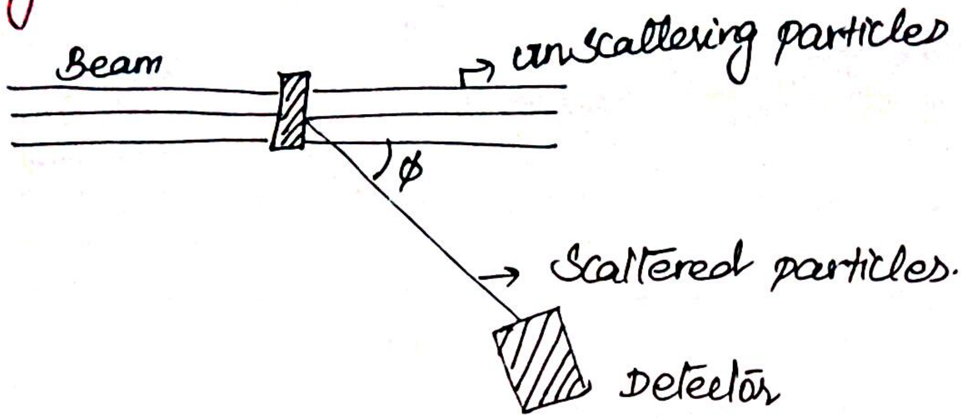
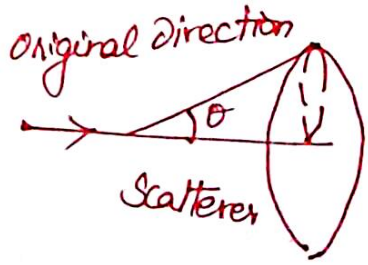
Scattering Cross Section (σ):



* The probability that a particle will be scattered as it transverses a given thickness of matter dx can be expressed in terms of a quantity called the scattering cross section.

* If n is the number of particles scattered / unit time into a solid angle $d\omega$ located in the direction (θ, ϕ), the n is directly proportional to the incident current.

$$n \propto J\, d\omega$$



$$n = \Sigma(\omega) J\, d\omega$$

$\Sigma(\omega) \rightarrow$ Constant of proportionality which has dimension of surface area.

Differential Scattering cross Section:

→ Let us now consider the target to be made up of a large number N of atomic / nuclear Scattering centres & the distance between these atoms / nuclei are sufficiently large w.r.to wavelength of incident particles.

→ Then each Scattering centre acts as it were alone.
Moreover if target is sufficiently thin,

$$n \propto N J d\omega$$

$$\Rightarrow \sigma(\omega) N J d\omega$$

$\sigma(\omega)$ → constant of probability, has surface area of Sca. centre direction $\omega(\theta, \phi)$

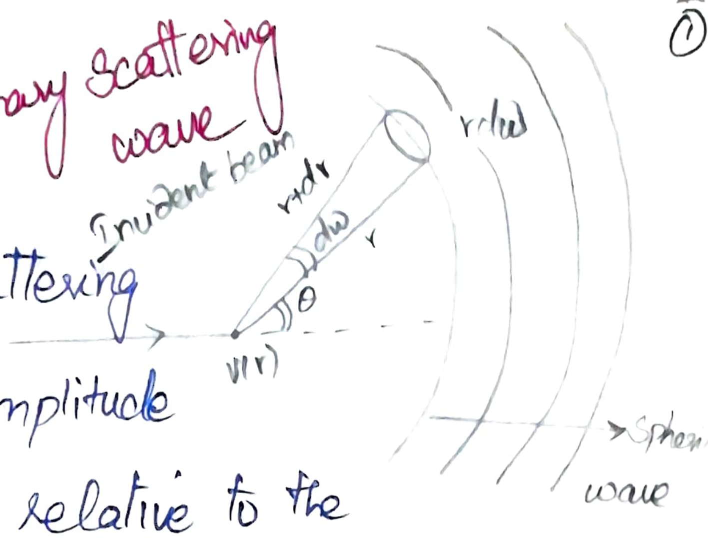
Total Scattering cross Section.

→ The total no. of particles Scattered/unit time is obtained by integrating n over all angles.

$$N_{total} = \int \sigma(\omega) N J d\omega = N J \sigma_{total}$$

$$\sigma_{total} = \int \sigma(\omega) d\omega.$$

Scattering Amplitude: Stationary Scattering wave



In quantum physics, the scattering amplitude is the probability amplitude of the outgoing spherical wave relative to the incoming plane wave in a stationary state scattering process.

* a quantity depending on the energy + sca. angle, which specifies the wavefn. of ◦les scattered in a collision

→ In wave mechanics, an incident beam of particles is represented by a plane wave in incident channel.

→ Let us consider the scattering of a particle of mass m, by central potential $V(r)$ such that $V(r) \to 0$; more rapidly than $\frac{1}{r}$ as $r \to \infty$.

* Incident beam is $\parallel^{el}$.
* Scattering potential is spherically sym.
* sca. ◦le is detected at a distance far away from the scatterer.

→ Let $E \to$ Energy ; $p = \hbar k \to$ the initial momentum of the ◦ where k - wave vector.

→ The Schroedinger eqn for central potential $V(r)$ is

$$V(r) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\varphi_k(r) = E\,\varphi_k(r) \quad \to (1).$$

→ The wave fn. φ_k may be written as a function of θ, ϕ + radial distance r between two particles,

$$\varphi_k = \varphi_k(r, \theta, \phi) \quad \to (2)$$

→ The Scattering is determined by the asymptotic form of $\psi_k(r,\theta,\phi)$ in region where $V=0$, (or) $r \to \infty$.

→ we want to represent it in two parts, → Incident wave
→ Scattered radially outgoing wave.

$$\psi_k(r,\theta,\phi) = \lim_{r\to\infty} \underbrace{e^{ikr}}_{\text{Incident}} + \underbrace{f(\omega)\frac{e^{ikr}}{r}}_{\text{scattered}}$$

→ Let us assume that one solution of this type exists for each value of k.

→ The two terms of asymptotic form are easily interpreted if we remember the definition of current density vector,

$J = I/A$ $\frac{\text{current}}{\text{Area}}$ $$J = \frac{\hbar}{2im}(\psi^* \nabla\psi - \psi\nabla\psi^*) \quad \to (3).$$

→ The plane wave term $e^{ik.r}$ represent a wave of unit density. & of current density $\hbar k/m$.

J

→ Retaining the only lowest order in r,
the term $\frac{f(\omega)}{r}e^{ikr}$ represent a wave of density $\frac{|f(\omega)|^2}{r^2}$ & $\frac{|f(\omega)|^2}{r^2}\frac{\hbar k}{m}$ as current density directed along ω direction towards increasing r

→ In accordance with this interpretation we can calculate the no. of ϕ'les emitted / unit time into solid angle $d\omega$ located in the direction ω.

→ The scattering wave fn. is $f(\omega)\frac{e^{ik.r}}{r}$ & Hence

→ Density of scattered particles $\rho_s = \left| f(\omega)\frac{e^{ikr}}{r}\right|^2$

$$\rho_s = \frac{1}{r^2}|f(\omega)|^2 \rightarrow (4)$$

→ from fig ; Small elementary area $= r.rd\omega = r^2 d\omega$.

→ The volume element bet. r & $r+d\omega = r^2 d\omega . dr$.

→ As ρ_s in the no. of scattered ϕ'le / unit volume, the no. of ϕ'le in elementary volume,

No. of scattered particle $N_s = \rho_s r^2 d\omega . dr$.

Subs., the value of ρ_s from eqn. (4) we get.

$$\therefore N_s = \frac{|f(\omega)|^2}{\cancel{r^2}} \cancel{r^2} d\omega . dr = |f(\omega)|^2 dr\, d\omega.$$

$$N_s = |f(\omega)|^2 dr\, d\omega \rightarrow (5)$$

→ The no. of scattered particles/unit time,

$r(t)$ → position fn. (vector component)
↳ moves with object
$V(t)$ → velocity fn.

$$V(t) = \frac{dr}{dt}$$

$$\frac{dN_s}{dt} = |f(\omega)|^2 d\omega \cdot \frac{dr}{dt}$$

$V(t) = dr/dt$

$$= |f(\omega)|^2 d\omega \cdot V$$

$$= |f(\omega)|^2 d\omega \cdot \frac{\hbar k}{m}$$

$$\left[\because \underset{\text{Momentum}}{P} = \hbar \underset{\text{wave number}}{k} \quad J = \frac{\hbar k}{m} \; ; \quad J = V \right]$$

$P = mV$ $\therefore V = \hbar k/m$.

$$\therefore \frac{dN_s}{dt} = |f(\omega)|^2 \frac{\hbar k}{m} d\omega \rightarrow (6)$$

→ If J is current density $J = \underset{\text{charge density}}{\rho} V \rightarrow (7)$; but $\rho = 1$ for incident particle

$$\therefore J = V = \frac{\hbar k}{m} \rightarrow (8)$$

→ Also $\sigma(\omega)$ is differential scattering cross section, then no. of particles scattered in solid angle $d\omega$/unit time,

$$= J\,\sigma(\omega)\,d\omega = \frac{\hbar k}{m} \cdot \sigma(\omega)\,d\omega \rightarrow (9)$$

comparing eqn. (6) & (9) we get,

$$\frac{\hbar k}{m} \sigma(\omega)\,d\omega = |f(\omega)|^2 \frac{\hbar k}{m} d\omega$$

Diff. sca. cr. section = (Sca. ampl.)^2

$$\sigma(\omega) = |f(\omega)|^2 \rightarrow (10) \text{ Scattering amplitude}$$

Hence the total sca. cross section is

$$\sigma_{total} = \int |f(\omega)|^2 d\omega \rightarrow (11).$$

Green's Function:

✓ The Schroedinger equation for central potential $V(r)$ is written as,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi \quad \rightarrow (1)$$

✓ The complete time dependent solution,

$$\psi(r,t) = \psi(r).e^{-iEt/\hbar}$$

$$\psi(r,t) = [\underbrace{e^{ikr}}_{\text{Incident}} + \underbrace{\psi_s(r)}_{\text{Scattered wave}}]e^{-iEt/\hbar} \quad \rightarrow (2)$$

where,

$$\psi_s(r) = \underbrace{f(\theta,\phi).\frac{e^{ikr}}{r}}_{\text{Outgoing Sca}} + \underbrace{g(\theta\phi)\frac{e^{-ikr}}{r}}_{\text{Incoming Sca.}} \quad \rightarrow (3)$$

✓ The stationary state soln. of Sch. eqn (1) is

$$\psi(r) = e^{ikr} + \psi_s \quad \rightarrow (4)$$

finding changes in discrete energy levels

$H\psi = E\psi$

✓ comparing eqn (1) with $(H^0 + H')\psi = E\psi$; Perturbation operator is $V(r)$ ↓ disturbance

$V(r) << E$, unperturbed Hamiltonian

✓ ∴ The unperturbed Schroe. eqn is $\left(-\frac{\hbar^2}{2m}\nabla^2 - E\right)e^{ikr} = 0 \quad \rightarrow (5)$

✓ So that Schroe. eqn is now written as from eqn (1)

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - E\right]\psi_s = -V(r)\left[e^{ikr} + \psi_s\right]$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - E\right]\psi_s = -V(r)\,\psi(r) \rightarrow (6)$$

This eqn can be written as, $(\nabla^2+k^2)\psi_s = \frac{2m}{\hbar^2}V(r)\,\psi(r)$ $\rightarrow (7)$

where $k = \frac{2mE}{\hbar^2}$ (−) sign throughout multiply

Further substituting ; $\frac{2m}{\hbar^2}V(r)\,\psi(r) = -4\pi\rho(r) \rightarrow (8)$.

$\therefore$ eqn (7) can be, $(\nabla^2+k^2)\psi_s = -4\pi\rho(r) \rightarrow (9)$

✓ The quantity $\rho(r) \rightarrow$ source density for divergent spherical wave.

✓ using the principle of superposition eqn (9),

$\psi_{s_1}, \psi_{s_2} \rightarrow$ Solution of eqn ; $\rho_1(r) + \rho_2(r) \rightarrow$ density fn's

$\psi_{s_1} \simeq f_1\frac{e^{ikr}}{r}$; $\psi_{s_2} \simeq \frac{e^{ikr}}{r}$; then $\psi_s = \psi_{s_1} + \psi_{s_2}$ is a $\rightarrow (10)$ Solution of eqn (9) belonging to $\rho(r) = \rho_1(r) + \rho_2(r)$ such that

$\psi_s = f\frac{e^{ikr}}{r}$ where $f = f_1 + f_2 \rightarrow (11)$

$\rho(r) = \int \delta(r-r')\rho(r')\,dr' \rightarrow (12)$; $\rho(r) \rightarrow$ arbitary density
$\delta(r-r') \rightarrow$ point sources at point r'.

✓ Now in order to express ψ_s as a fn. of $\rho(r)$ we make use of the following theorem.

$$(\nabla^2+k^2)\,G(r,r') = -4\pi\delta(r-r') \rightarrow (13)$$

where $G(r,r') = \frac{\exp(ik|r-r'|)}{|r-r'|} \rightarrow$ Green's fn.

✓ If $G(r,r')$ is asymptotic to a fn. of r of the form (11), then the solution of scattering problem for the density $\rho(r)$ is

$$\psi_s = \int G(r,r')\,\rho(r')\,d\tau' \rightarrow (14)$$

✓ It may be noted that Green's fn,

$$G(r,r') = \frac{\exp[ik|r-r'|]}{|r-r'|} \rightarrow (15)$$

✓ Green's fn. is a solution of the scattering problem for a source of unit strength at point r. ⟹

* To prove this it must be shown that the eqn (13) is satisfied and the solution has proper asymptotic form.

* First we change the origin of coordinates to point r', so that eqn (13) takes the form,

$$(\nabla^2+k^2)\,G(\underset{(r,r')}{r}) = -4\pi\,\delta(\underset{(r-r')}{r}) \rightarrow (16)$$

where $G(r) = \dfrac{e^{ikr}}{r} \rightarrow (17)$; r being radial distance.

✓ we notice by direct differentiation that if $r \neq 0$

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = 0 \quad \rightarrow (18)$$

∴ eqn (16) is satisfied in every region which does not contain the source point.

✓ To prove the singularity at $r=0$ is properly represented by G,

$$\delta(r) = \frac{1}{4\pi}(\nabla^2 + k^2)G(r) \quad \rightarrow (19) \rightarrow$$ satisfy the delta function

✓ From eqn (16) & (18) we note that if $\delta(r) = 0$ if $r \neq 0$

✓ Thus $\delta(r)$ satisfies the I requirement of delta fn, namely that it is zero everywhere except at $r=0$.

✓ The another condition of delta fn is

$$\int_\tau \delta(r)F(r)d\tau = F(0) \quad \rightarrow (20)$$

where $F(r)$ is any continuous function of r which has value $F(0)$ at the origin & τ represent the region of integration.

✓ Let us choose for our range of integration a small sphere of radius ϵ. Consider the identity.

eqn (16) $\quad \int u\,dv - v\,du$

$$\int_{\tau}\left[\left(\nabla^2 G + k^2 G\right)F - \left(\nabla^2 F + k^2 F\right)G\right]d\tau = \int_{S}\left[\frac{\partial G}{\partial r}F - \frac{\partial F}{\partial r}G\right]ds \quad \to (21)$$

where S represent the surface of the sphere τ.

✓ If $F(r)$ is sufficiently regular within the sphere τ, then we may assume positive numbers M and N such that

$$\left|\nabla^2 F + k^2 F\right| < M \ \ \& \ \ \left|\frac{\partial F}{\partial r}\right| < N \ (r \le \epsilon) \quad \to (22)$$

✓ In other words the above fn. are bounded in τ

$$\left|\int_{\tau}\left(\nabla^2 F + k^2 F\right)G\,d\tau\right| \le M\int_{\tau}|G|\,d\tau$$

$$= M\int_0^{\epsilon}\frac{1}{r}4\pi r^2\,dr = 2\pi M\epsilon^2 \quad \to (23)$$

$$\left|\int_S \frac{\partial F}{\partial r}G\,ds\right| \le N\int_S |G|\,ds = N\left[\frac{1}{r}\cdot 4\pi r^2\right]_0^{\epsilon}$$

$$'' \qquad = 4\pi N\epsilon \quad \to (24)$$

∴ If we take the limit $\epsilon \to 0$,

$$\int_{\tau}\left(\nabla^2 F + k^2 F\right)G\,d\tau \to 0 \ \ \& \ \ \int_S \frac{\partial F}{\partial r}G\,ds \to 0$$

✓ In view of above relations eqn. (21) in limit $\epsilon \to 0$ yields,

$$\lim_{\epsilon \to 0} \int_{\tau} (\nabla^2 G + k^2 G) F d\tau = \lim_{\epsilon \to 0} \int_S \frac{\partial G}{\partial r} F ds$$

$$= \lim_{\epsilon \to 0} \int_S \left[ik \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right] F ds.$$

$$= \lim_{\epsilon \to 0} 4\pi\epsilon^2 F(0) \left\{ ik \frac{e^{ik\epsilon}}{\epsilon} - \frac{e^{ik\epsilon}}{\epsilon^2} \right\}$$

$$\lim_{\epsilon \to 0} \int (\nabla^2 G + k^2 G) F . d\tau = -4\pi F(0) \quad \to (25)$$

✓ Because of relation (18) the only contribution to the integral (20) must be noted to singularity of $\delta(r)$ at $r = 0$ & by eqn (25).

$\to 0$ at eqn (18)

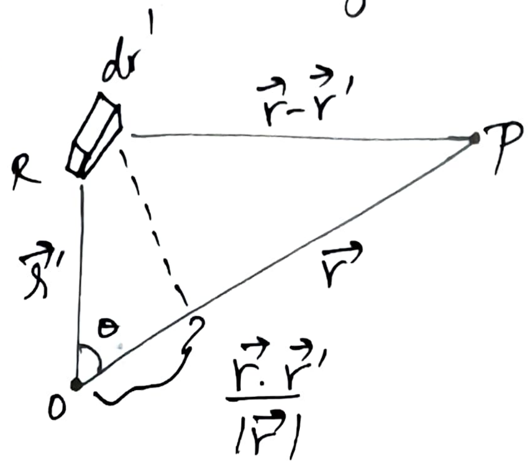
$$\frac{1}{4\pi} \int_{\tau} (\nabla^2 + k^2) G F(r) d\tau = F(0)$$

$$\int_{\tau} \delta(r) F(r) d\tau = F(0)$$

Hence $\delta(r)$ is a delta fn. Returning to the original co-ordinate system by substitution $r \to r + r'$, we obtain eqn (13) instead of (16)

4

The vectors of r & r'



✓ If $|r|$ is large compared to $|r'|$, then it is clear that

$$|r-r'| = r - r'\cos\theta$$

$$= r - \cancel{r'}\cdot\frac{r.r'}{r\cancel{r'}} = r - \frac{r.r'}{r}$$

$$|r-r'| \simeq |r| - \frac{r'.r}{|r|}$$

$$\underline{|r-r'|} = r - \frac{r'.r}{r} \rightarrow (26)$$

∵ $|r|$ is sufficiently large,

Substituting value of $|r-r'|$ from eqn (26) in (15) we get

$$G(r,r') = \frac{\exp(ik\,\underline{|r-r'|})}{|r-r'|}$$

$$= \frac{\exp\left[ik\left(r-\frac{r'.r}{r}\right)\right]}{\left(r-\frac{r'.r}{r}\right)} \qquad (1-x)^{-n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$= \frac{\exp\left[ik\left(r-\frac{r'.r}{r}\right)\right]}{r}\left[1+\frac{r'.r}{r^2}+\cdots\right]$$

Thus $r \to \infty$ we obtain.

$$G(r,r') \approx \exp\left(-ik\frac{r'.r}{r}\right) \cdot \frac{e^{ikr}}{r}$$

writing $k' = \frac{kr}{r}$,

$$G(r,r') \cong e^{-ik'.r'} \cdot \frac{e^{ikr}}{r} \quad \rightarrow (27)$$

The first term in this expression depends upon the orientation of k' relative to vector r'. ie it is of the form $f(\theta,\phi)$

✓ Thus eqn (27) represent the asymptotic form of Green's fn & represent an outgoing wave.

✓ Thus the Green's fn represent the solution of the Scattering problem of unit intensity at point r'.

Setting,

$$\rho(r') = -\frac{1}{4\pi}\frac{2m}{\hbar^2} V(r')\psi(r')$$

from eqn (8) & the value of $G(r,r')$ from eqn (15), eqn (14) gives,

$$\psi_s = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int V(r')\psi(r')\frac{e^{ik(r-r')}}{|r-r'|}d\tau' \quad \rightarrow (28)$$

✓ All these Spherical waves are compounded at the point r which is then added to the incident wave to produce the total wave function ψ of r. [ie., $\psi(r)$]

Thus,

$$\psi(r) = e^{ik'.r'} + \psi_s$$

$$= e^{ik'.r'} - \frac{m}{2\pi\hbar^2}\int v(r')\, \frac{e^{ik|r-r'|}}{|r-r'|}\, d\tau' \rightarrow (29)$$

✓ If the potential energy function is confined to a limited region of space, then the asymptotic form of Green's function ~~from~~ (27) can be substituted in eqn (14),

$$\psi_s = -\frac{m}{2\pi\hbar^2}\int G(r,r')\, v(r')\psi(r')\, d\tau'$$

$$\psi_{s\,(r\to\infty)} = -\frac{m}{2\pi\hbar^2}\,\frac{e^{ikr}}{r}\int \underbrace{e^{-ik'.r'}\, v(r')\psi(r')\, d\tau'}_{(\theta,\phi)} \rightarrow (30)$$

where as $\psi_s = f(\theta,\phi)\,\frac{e^{ikr}}{r}$.

$\therefore$ Scattering amplitude $f(\theta,\phi)$ is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-ik'.r'} V(r')\psi(r')\,d\tau' \quad \rightarrow (31)$$

This eqn yields for the Scattering cross section,

Diff. Cross Section = Scattering Amplitude^2

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2$$

$$= \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int e^{-ik'.r'} V(r')\psi(r')\,d\tau'\right|^2$$

$\rightarrow (32)$.

Born Approximation:

* Max Born who proposed this approximation in early days of quantum theory development.

* It is the perturbation method applied to scattering by an extended body.

* It is accurate if the scattered field is small compared to the incident field on the scatterer.

* It can be used to evaluate the scattering amplitude $f(\theta,\phi)$ and hence the differential cross section $\sigma(\theta,\phi) = |f(\theta,\phi)|^2$

* If the scattering takes place from the scattering centres which are localized but are weak, such that scattering does not take place at large distance from the scatterer & scattered wave is weak.

* Born approximation is applicable whenever potential fn. V is fairly small.

" The Born-approxi. simply accounts neglecting the rescattering of scattered waves provided the scattered wave is small "

✓ The total wave function of the scattering problem with Source point at r'' is given by,

$$\psi(r) = e^{ik.r} - \frac{m}{2\pi\hbar^2}\int G(r,r'')\, v(r'')\, \psi(r'')\, d\tau'' \rightarrow ①$$

Replacing r by r' in above eqn, we get

$$\psi(r') = e^{ik.r'} - \frac{m}{2\pi\hbar^2}\int G(r',r'')\, v(r'')\, \psi(r'')\, d\tau''$$

Subs this $\psi(r')$ in the integral of equation representing the total wave fn. with source point at r' viz.,

$$\psi(r) = e^{ik.r} - \frac{m}{\hbar^2 . 2\pi}\underbrace{\int G(r,r')\, v(r')\, \psi(r')\, d\tau'}_{x} \rightarrow ②$$

(a-b)

$x(a-b) = ax+bx$

we get.

$$\psi(r) = e^{ikr} - \frac{m}{2\pi\hbar^2}\int G(r,r')\, v(r') . \left[e^{ikr'} - \frac{m}{2\pi\hbar^2}\int G(r',r'')\, v(r'')\, \psi(r'')\, d\tau''\right] . d\tau'$$

ax

bx

$$+ \left(\frac{m}{2\pi\hbar^2}\right)^2 \iint G(r,r')\, v(r')\, G(r'r'')\, v(r'')\, d\tau'' . d\tau' \rightarrow ③$$

This eqn. obtained by the process of iteration is called first iterated form of eqn. ②. This process can be repeated indefinitely resulting is an infinite (Neumann) series which can be expected to represent a solution provided the series converges.

✓ I e^{ikr} represents the incident wave function.

Remaining terms correspond to scattered wave function.

✓ The I term in the scattered wave represents single scattering of the incident wave $\exp(ik.r')$ by interaction $V(r')$ in volume element dr'.

✓ This produces a wave which travels from r' to pt. of observation r & total wave arising from single scattering is obtained by integration over the region in which force is effective.

✓ II term incident wave $e^{ik.r''}$ is scattered at pt. r'', $[V(r'').e^{ikr''}]$ travels to pt. r, $[G(r,r'')V(r'')e^{ik.r''}]$ where it is again scattered & travels from r' to r.

✓ The total effect of all such scattering is obtained by integration over r'' and r'.

✓ Accordingly n^{th} term represent the contribution of waves which have been scattered n times in the region of interaction before travelling to pt. r, where total contribution is observed.

✓ Thus the first Born approxi: $G(r,r') = e^{-ik r'} \cdot \frac{e^{ikr}}{r}$.

$$\psi = e^{ik.r} - \frac{m}{2\pi\hbar^2}\int G(r,r')\, v(r')\, e^{ik.r'}\, d\tau' \rightarrow (4)$$

✓ Thus n^{th} term is called ~~E~~ n^{th} term called n^{th} Born approxi.

$$\psi = e^{ikr} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-ik'.r'}\, v(r')\, e^{ik.r'}\, d\tau' \rightarrow (5)$$

But $$\psi = e^{ik.r} + f(\theta,\phi)\frac{e^{ikr}}{r}. \rightarrow (6)$$

∴ I Born approximation scattering amplitude,

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(k-k').r'}\, v(r')\, d\tau' \rightarrow (7)$$

Hence scattering cross section,

$$\sigma(\theta,\phi) = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int e^{i(k-k').r'}\, v(r')\, d\tau'\right|^2 \rightarrow (8)$$

Condition for validity of Born Approximation.

→ The Born approxi. is valid only whenever the total wave fn. & incident wave fn. e^{ikr} are nearly same.

→ ∴ the scattered wave $\psi_s(r)$ is small compared to e^{ikr} in the region where $V(r)$ is large.

→ In most cases both $V(r)$ & $\psi_s(r)$ are largest near origin, so that rough criterion for validity of Born approxi. is

$$|\psi_s(r)|^2 < 1 \quad \text{for small values of } r \rightarrow (1)$$

→ In case $\psi_s(r)$ is small when r is small but large for intermediate values of r such that $V(r)$ is appreciable, we must carefully apply this criterion.

→ Having $\psi_s(r)$ small everywhere provides sufficient condi. for validity of Born approxi. but not a necessary condition

→ The total wave function will not differ greatly from the initial wave fn. If the phase of the incident wave is not much altered as it passes through the region in which it is influenced by the perturbing potential.

→ At great distance the magnitude of wave vector is

$$K = \frac{\sqrt{2mE}}{\hbar}$$ & Centre of force is $\frac{\sqrt{[2m(E-V)]}}{\hbar}$

→ The change of phase due to the potential is then given by,

$$\Delta\phi = \int_0^\infty \sqrt{\left(\frac{2m}{\hbar^2}\right)} \left[\sqrt{(E-V)} - \sqrt{E}\right] dr \quad \rightarrow (2).$$

→ ∵ the difference is small compared to unity, we may take it as an indication that the wave fn. is not very different from that in the absence of potential.

→ Thus the first order Born approxi. will be valid if

$$|\Delta\phi| = \left| \sqrt{\left(\frac{2m}{\hbar^2}\right)} \int_0^\infty \left[\sqrt{(E-V)} - \sqrt{E}\right] dr \right| << 1 \quad \rightarrow (3)$$

Partial wave Analysis:

→ PWA refers to a technique for solving scattering problems by decomposing each wave into its constituent angular -momentum components and solving using boundary conditions.

⇒ It is based upon an expansion of the wave function in terms of angular momentum eigen functions.

⇒ It is applicable to spherically symmetric potentials, & useful only for low energy incident particles.

✓ Consider a plane wave incident along z-axis in a region having interaction potential function $V(r)$.

✓ Then total wave fn. may be expressed as

$$\psi(r) = \underbrace{e^{ikz}}_{\text{Incident wave}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\text{Scattered wave}} \longrightarrow (1)$$

where eqn (1) gives the solution of 3D Schroedinger equation

$$\nabla^2\psi + \frac{2\mu}{\hbar^2}\left[E - V(r)\right]\psi = 0 \longrightarrow (2)$$

✓ The Solution of Eq (2) may be expressed as.

$$\psi(r) = \sum_{l=0}^{\infty} R_l(r)\, Y_{l0}^{(\theta)} \longrightarrow (3)$$

✓ This is Superposition of a number of waves. Each term in the above eqn is called a partial wave, corresponding to a particular value of l.

✓ The fn. $R_l(r)$ are called radial wave fuction.

Put $\frac{2\mu E}{\hbar^2} = k^2$ and $\frac{2\mu V(r)}{\hbar^2} = U(r)$

∴ Eqn. (2) becomes,

$$\nabla^2\psi + [k^2 - U(r)]\phi = 0 \longrightarrow (4)$$

✓ As there is symmetry about polar axis ie, z axis $(m=0)$ & Potential Energy fn. does not involve ϕ, therefore the Soln. of eqn (4) may be expressed as,

$$\psi(r,\theta,\phi) = \phi(r,\theta) \sum_l R_l(r) P_l(\cos\theta) \longrightarrow (5)$$

Put $\chi_l = rR_l(r) \longrightarrow (6)$

∴ Eqn (5) becomes,

$$\phi(r,\theta) = \sum_l r^{-1}\chi_l(r) P_l(\cos\theta) \longrightarrow (7)$$

Here $\chi_l(r)$ satisfies the eqn,

$$\frac{d^2\chi_l}{dr^2} + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \rightarrow (8)$$

In the asymptotic region, U & l terms in Eq (8) may is neglected in comparision with k^2. So, the eqn. (8) becomes,

$$\frac{d^2\chi_l}{dr^2} + k^2\chi_l = 0 \rightarrow (9)$$

$$\chi_l = e^{\pm ikr} \rightarrow (10)$$ which gives the radial wave only.

For better approximation, we define a distance "a" such that when $r > a$; $V(r) = 0$ & when $r < a$; $V(r)$ is a finite.

Then eqn 8 becomes,

$$\frac{d^2\chi_l}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \rightarrow (11)$$

which is spherical Bessel equation.

* on Solving Eqn. (11) we get.,

$$R_l(r) = \frac{A_l}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \rightarrow (12)$$

Phase Shift in l^{th} partial wave.

* Eqn. (5) becomes,

$$* \ \psi(r,\theta) = \sum_l \frac{A_l}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) P_l \cos\theta$$

(or)

$$* \ \psi(r,\theta) = \sum_l \frac{A_l}{kr} \left\{ \frac{e^{i\left(kr - \frac{l\pi}{2} + \delta_l\right)} - e^{-i\left(kr - \frac{l\pi}{2} + \delta_l\right)}}{2i} \right\} P_l(\cos\theta) \rightarrow (13)$$

* This eqn. is identical with asymptotic form of Eq. (1)., i.e.,

$$\psi(r) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

spherical Bessel fn.

$$\rightarrow \sum_{l=0}^{\infty} (2l+1) i^l \left(\frac{1}{kr}\right) \sin\left(kr - \frac{1}{2}l\pi\right) P_l(\cos\theta) \rightarrow (14)$$

$$\therefore\ \psi(r) = \sum_l (2l+1) i^l \cdot \frac{1}{kr} \underline{\sin\left(kr - \frac{l\pi}{2}\right)} P_l(\cos\theta) + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\text{Sca. wave}}$$ (3)

$$\psi(r) = \sum_l (2l+1) i^l \cdot \frac{1}{kr}\left[\frac{e^{i\left(kr - \frac{l\pi}{2}\right)} - e^{-i\left(kr - \frac{l\pi}{2}\right)}}{2i}\right] P_l(\cos\theta) + r^{-1} f(\theta) e^{ikr}$$

$\rightarrow$ (15)

Eqn. (13) & (15) represent the same fn. & hence the co.efficients of e^{ikr} and e^{-ikr} should be identical.

Equaling the co-efficients gives us,

$$\sum_l A_l \frac{e^{i\left(-\frac{l\pi}{2} + \delta_l\right)}}{2ikr} P_l(\cos\theta) = \sum_l \frac{(2l+1) i^l e^{-il\pi/2}}{2ikr} \cdot P_l(\cos\theta) + r^{-1} f(\theta)$$

$\rightarrow$ (16)

and

$$\sum_l A_l \frac{e^{i\left(-\frac{l\pi}{2} + \delta_l\right)}}{2ikr} \times P_l(\cos\theta) = \sum_l \frac{(2l+1)}{2ikr} i^l e^{il\pi/2} \cdot P_l(\cos\theta)$$

$\rightarrow$ (17)

✓ From eqn. (17) we get;

$$A_l = (2l+1) i^l e^{i\delta_l} \rightarrow (18)$$

✓ Substituting this value of A_l in Eqn. (16), we get,

$$\sum_l \frac{(2l+1) i^l e^{i\delta l} e^{i\left(-l\frac{\pi}{2}+\delta_l\right)}}{2ikr} P_l \cos\theta = \sum_l \frac{(2l+1) i^l e^{-il\frac{\pi}{2}}}{2ikr} \cdot P_l(\cos\theta) + \frac{1}{r} f(\theta)$$

* Put $i^l = e^{il\pi/2}$

$$f(\theta) = (2ik)^{-1} \sum_{l=0}^{\infty} (2l+1)\left(e^{2i\delta_l} - 1\right) P_l(\cos\theta)$$

(or) Scattering Amplitude

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta) \rightarrow (19)$$

✓ This gives $f(\theta)$ as a sum of contributions from Partial waves with $l = 0$ to ∞.

✓ The differential scattering cross section,

$$\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) e^{i\delta_l} \sin\delta_l \right|^2 \rightarrow (20)$$

* The total Scattering Cross section,

$$\sigma = 2\pi \int \sigma(\theta) \sin\theta \, d\theta$$

$$\sigma_{Total} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \rightarrow (21)$$

Phase shift of l^{th} partive wave

Phase Shifts - Low energy Scattering:

✓ The scattering cross section vanishes for $\delta_l = 0/180^\circ$ & the cross section is maximum if the value of

$\delta_l = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$ etc.,

✓ According to the eqn, $R_l(r) = \frac{A_l}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$, we have,

$$R(r) = \lim_{r \to \infty} \frac{A_l}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$

$\left[\because \text{It comes from } R(r) \to A j_l(kr)\right]$

✓ So δ_l is the difference in phase between the asymptotic form of the actual radii function $R(r)$ and the radial function $j_l(kr)$ in the absence of scattering potential i.e., $V=0$. $j_l(kr)$ will be maximum when $r \simeq l/k$ hence for the value of r.

* we choose "a"; $r \approx a \simeq l/k$ we get higher phase difference.

Small phase shift :

✓ The phase shift will be very small if $a << l/k$. Thus the summation $\sum_{l=0}^{\infty}$ involves the summation of few terms such as $\sum_{l=0}^{l=ak}$.

Calculation of δ_l :

* δ_l is calculated by applying boundary condition for the continuity of R_l at $r=a$ in the region $r<a$ and $r>a$

$$\left.\left(\frac{1}{R_l}\frac{dR_l}{dr}\right)_{r<a}\right|_{\text{at } r=a} = \left.\left(\frac{1}{R_l}\frac{dR_l}{dr}\right)_{r>a}\right|_{\text{at } r=a}.$$

* But $R_l = A_l\left[\cos\delta_l\, j_l(kr) - \sin\delta_l\, \eta_l(kr)\right]$

$$\therefore \left.\left(\frac{1}{R_l}\frac{dR_l(r)}{dr}\right)_{r>a}\right|_{\text{at } r=a.} = \left. k\left[\frac{\cos\delta_l\, j_l'(ka) - \sin\delta_l\, \eta_l'(ka)}{\cos\delta_l\, j_l(ka) - \sin\delta_l\, \eta_l(ka)}\right]\right|_{r>a}$$

✓ Let $\frac{1}{R_l}\frac{dR_l}{dr}\, r<a \Big|$ at $r=a=\gamma_l$.

÷ by $\cos\delta$;

$$\gamma_l = \frac{k\left[j_l'(ka) - \tan\delta_l\, n_l'(ka)\right]}{j_l(ka) - \tan\delta_l\, n_l(ka)}$$

$\gamma_l\left[j_l(ka) - \tan\delta\, n_l(ka)\right]$

$= k\left[j_l'(ka) - \tan\delta\, n_l'(ka)\right]$

$k(j_l'(ka) - \gamma_l(j_l(ka))) =$

$k\tan\delta\, n_l'(ka) - \gamma_l \tan\delta\, n_l(ka)$

ten $$\tan\delta_l = \frac{k j_l'(ka) - \gamma_l j_l(ka)}{k n_l'(ka) - \gamma_l n_l(ka)}$$

where,
$$j_l'(ka) = j_{l-1}(ka) - \frac{l+1}{ka} j_l(ka) \text{ and}$$
$$n_l'(ka) = n_{l-1}(ka) - \frac{l+1}{ka} n_l(ka)$$

Here, $\gamma_l \rightarrow$ ratio of slope of interior wavefunction.

✓ Eqn. (35) can be used at once to obtain an approxi. expression for δ_l where l is large & δ_l ~~when~~ is expected to be small.

✓ $$\gamma_l = k\left[\frac{j_l'(ka)}{j_l(ka)} + \varepsilon_l\right]$$

$$|\varepsilon_l| < \left|\frac{j_l'(ka)}{j_l(ka)}\right| \longrightarrow (36)$$

✓ Eqn (25) Can be written by charging j_l when $l > (ka)^2$ and use the value of j_l in terms of Sine & Cosine, the inequality (36) becomes,

$$|\mathcal{E}_l| < \frac{1}{ka}$$

✓ Eqn (37) may be approxi,

$$\delta_l = \frac{\mathcal{E}_l (ka)^{2l+2}}{[(2l+1)!]^2} = -\frac{\mathcal{E}_l 2^{2l} (l!)^2 (ka)^{2l+2}}{[(2l+1)!]} \rightarrow (38)$$

By using Stirling's formula, we get.,

$$\log|\delta_l| \propto \log|\mathcal{E}_l| = 2l[\log(ka) + 1 + \log 2] - 2l \log l.$$

✓ The following are explanatory remarks concerning δ_l.

(i) From earlier eqn., an attractive field $\phi(r)$ is shifted outward relative to other fn.,

$\delta_l >$ for attractive field,

$\delta_l < 0$ for repulsive field.

(ii) For large k and l, the phase shift can be calculated by the Born approximation it becomes,

$$\delta_l \simeq -\frac{1}{2k} U(r_0) r_0.$$

$$\delta_l \approx -\frac{1}{2k} U(r_0)\, r_0.$$

where r_0 is the classical distance of closest approach.

✓ For large l, r_0, $\rho \approx$ impact parameter. The series for total cross section behaves like,

$$\sigma \propto \sum (2l+1)\delta_l^2 = \frac{1}{4}\int_0^\infty d\rho \cdot \rho^3 U^2(\rho)$$

$$p\rho = \hbar k\rho.$$

In order that it may coverge $U(\rho)$ must decrease with distance faster than $1/r^3$

(iii)

For low energy scattering by a potential of the asymptotic for c/r^n, the phase shift variations for various l are,

$$\delta_l \propto k^{2l+1} \qquad \text{for } 2l < n-3;$$

$$\delta_l \propto k^{2l+2} \log k. \qquad \text{for } 2l = n-3;$$

$$\delta_l \propto k^{n-2} \qquad \text{for } 2l > n-3.$$

Scattering length and Effective range theory for Low Energy Scattering.

✓ The method of partial waves is of special interest at low energies if the energy of the incident beam is so low that $ka < 1$ where $a \rightarrow$ range of potential, then the only $l = 0$ (or) S-wave is given by,

$$f(\theta) = \frac{1}{k} e^{i\delta_0} \sin\delta_0 \quad \rightarrow (1)$$

As the scattering amplitude is independent of θ & ϕ,

$\therefore$ Total scattering cross section,

$$\sigma_{Total} = \frac{4\pi}{k^2} \sin^2\delta_0 \quad \rightarrow (2)$$

In the limit $k \rightarrow 0$, $\sigma_{total} \Rightarrow \sigma_0$ which is called the low-energy cross section.

Then,

$$\lim_{k \to 0} k^2 \operatorname{cosec}^2 \delta_0 = \frac{4\pi}{\sigma_0} \; ;$$

$$\lim_{k \to 0} \sin\delta_0(k) \rightarrow 0 \; ; \; \delta(k) \propto 0 \text{ or } \pi \text{ in zero} \rightarrow \text{energy limit.}$$

(i.e) $\delta_{(k)}$ approaches 0 or π; in zero energy limit.

✓ It is found that low energy cross section can be described instead of δ_0, by two quantities that characterizes $V(r)$ completely as far as low energy scattering is concerned. These are the "effective range" r_0 and the "scattering length" 'a' introduced by Fermi in connection with nucleon-nucleon scattering.

✓ If the system has a bound state with a small binding energy, the two low energy parameters r_0 and a will be completely determined by the bound state wave function.

From (21a) and the $u_k(r)$ for $k \to 0$, $r \to \infty$ in (13), we get

$$u_0 = \left(1 - \frac{R}{a}\right)\frac{\sin \beta r}{\sin \beta R}\ ;\ 0 < r < R$$

and with (13), for v_0, we have from (12a)

$$r_0 = 2R - 2\frac{R^2}{2} + \frac{3R^{2R^3}}{3a^2} + \left(1 - \frac{R}{a}\right)^2\left(\frac{1}{\beta \tan \beta R} - \frac{R}{\sin^2 \beta R}\right) \quad ...(24)$$

If $\beta R = \left(n + \frac{1}{2}\right)\pi$, $n = 0, 1, 2, \ldots$; a stands to $\pm\infty$ and (24) simplifies and the total cross-section σ in terms of a and r_0 is

$$\sigma = \frac{4\pi a^2}{1 + a(a - r_0)k^2 + \left(\frac{1}{2} a r_0\right)k^4}.$$

13/15 SCATTERING BY A PERFECTLY RIGID SPHERE

A perfectly rigid sphere of radius a is represented by the potential

$$V(r) = \begin{cases} \infty & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

The wave function vanishes for $r < a$.

The Schoredinger wave equation (radial part) for $r > a\ [V(r) = 0]$ is

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2}\right] u_l(r) = 0$$

Substituting $\xi = kr$ or $r = \frac{\xi}{k}$, we have

$$\frac{d}{d\xi}\left(\xi^2 \frac{\partial u_l(\xi)}{\partial \xi}\right) + \left[\xi^2 - l(l+1)\right] u_l(\xi) = 0 \quad ...(1)$$

Further substituting

$$u_l(\xi) = \frac{v_l(\xi)}{\sqrt{\xi}}. \quad ..(2)$$

in (1,) we get

$$\xi^2 \frac{d^2 v}{d\xi^2} + \xi \frac{dv}{d\xi} + \left[\xi^2 - \left(l + \frac{1}{2}\right)^2\right] v_l = 0. \quad ...(3)$$

This is the Bessel's equation of order $\left(l + \frac{1}{2}\right)$. Its general solution is linear combination of Bessels function $J_{l+1/2}(\xi)$ and Neumann's function $N_{l+1/2}(\xi)$. The solution $N_{l+1/2}$ is not satisfactory since it diverges at $\xi = 0$; therefore the solution of (3) is expressed as

$$v_l = \sqrt{\left(\frac{\pi}{2}\right)} J_{l+1/2}(\xi)$$

where $\sqrt{\left(\frac{\pi}{2}\right)}$ is a constant chosen for conveience and it does not affect the solution of the problem.

$$u_l(\xi) = \frac{v_l(\xi)}{\sqrt{\xi}} = \sqrt{\left(\frac{\pi}{2}\right)} \frac{J_{l+1/2}(\xi)}{\sqrt{\xi}}$$

Now $u_l(\xi) = j_l(\xi)$ in the spherical Bessel's function, *i.e.*

$$j_l(\xi) = \sqrt{\left(\frac{\pi}{2\xi}\right)} J_{l+1/2}(\xi) \qquad \text{...(4)}$$

The spherical Neumann's functions are defined as

$$n_l(\xi) = (-1)^{l+1} \sqrt{\left(\frac{\pi}{2\xi}\right)} J_{-l-1/2}(\xi) \qquad \text{...(5)}$$

Therefore the general solution of (1) is expressed as

$$u_l = A\, j_l(\xi) + B\, n_l(\xi) \qquad \text{...(6)}$$

where A and B are constants.

$$\text{For } \xi \to \infty,\ J_{l+1/2} = \sqrt{\left(\frac{2}{\pi\xi}\right)} \sin\left(\xi - \frac{l\pi}{2}\right)$$

$$\therefore \quad \lim_{\xi \to \infty} j_l(\xi) = \sqrt{\left(\frac{\pi}{2\xi}\right)} [J_{l+1/2}(\xi)]_{\xi \to \infty} = \frac{\sin(\xi - l\pi/2)}{\xi} \qquad \text{...(7)}$$

Also

$$\lim_{\xi \to 0} n_l(\xi) = -\frac{\cos(\xi - l\pi/2)}{\xi} \qquad \text{...(8)}$$

Substituting $A = C\cos\delta_l$ and $B = -C\sin\delta_l$ in (6), we get

$$u_l = C\cos\delta_l\, j_l(\xi) - C\sin\delta_l\, n_l(\xi)$$

$$= C[j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l] \qquad \text{...(9)}$$

This equation represents the solution of wave equation for $r > a$. When $r = a$, the wave function vanishes,

i.e. $\quad u_l(r) = 0$ at $r = a$;

then equation (9) gives

$$0 = C[j_l(ka)\cos\delta_l - n_l(ka)\sin\delta_l]$$

or

$$\tan\delta_l = \frac{j_l(ka)}{n_l(ka)} \qquad \text{...(10)}$$

This equation gives phase shift δ_l for l^{th} partial wave.

From (10)

$$\sin^2\delta_l = \frac{j_l^2(ka)}{j_l^2(ka) + n_l^2(ka)} \qquad \text{...(11)}$$

$\therefore$ The *scattering cross-section for l^{th} partial wave* is given by

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l = \frac{4\pi(2l+1)}{k^2}\frac{j_l^2}{j_l^2 + n_l^2} \qquad \text{...(12)}$$

$\therefore$ The total scattering cross-section is

$$\sigma_{total} = \frac{4\pi}{k^2}\Sigma(2l+1)\frac{j_l^2}{j_l^2 + n_l^2} \qquad \text{...(13)}$$

From this equation the cross-section at all energies may be evaluated.

Now let us discuss the following two limiting cases :

(1) Low Energy Limit : The low energy limit implies the domain of k for which $ka << 1$, we have