# ELECTROSTATICS

## 1. Introduction:

### 1.1 Electrostatics – meanings

The branch of Physics tl at deals with static electricity, The Physics of electrostatic Phenomena. The branch of Physics which deals with static electric charges or charges at rest.

<u>Static Electricity</u>: The electricity produced by <u>friction</u> is called <u>frictional electricity</u>. If the charges in a body <u>do not move</u>, then, the frictional electricity is also known as <u>Static Electricity</u>.

Many problems in mechanics are greatly simplified by means of energy considerations. Hence, to know the mechanical behavior of an electrical system, the energy methods are to be studied. In general,

the energy of a system of charges, just like that of any other mechanical system, may be divided into its <u>potential</u> and <u>kinetic</u> <u>contributions</u>. Under Static conditions, the entire energy of the charge system exists as potential energy and, It is concerned particularly with the potential energy that arises from electrical interaction of the charges, the so-called "<u>electrostatic energy</u>".

### 1.2 <u>Coulomb's Law:</u>

All of electrostatics stems from the quantitative statement of Coulomb's law concerning the force acting between charged bodies at rest with respect to each other. Coulomb, in an impressive series of experiments, showed experimentally that the force between two small charged bodies separated in air a distance large compared to their dimensions Varies directly as the magnitude of each charge, Varies inversely as the square of the distance between them, is directed along the line joining the charges, and is attractive if the bodies are oppositely charged and repulsive if the bodies have the same type of charge.

The total force produced on one small charged body by a number of the other small charged bodies placed around it is the vector sum of the individual two-body forces of Coulomb. Strictly speaking, coulomb's conclusions apply to charges in vacuum or in media of negligible susceptibility. <u>Let $q_1$ and $q_2$ be two point charge placed in air at a distance r.</u> According to coulomb's law

$$F \leftarrow \overset{q_1}{\underset{|\text{-}\text{-}\text{-}\text{-}r\text{-}\text{-}\text{-}\text{-}|}{\bullet}} \overset{q_2}{\bullet} \rightarrow F$$
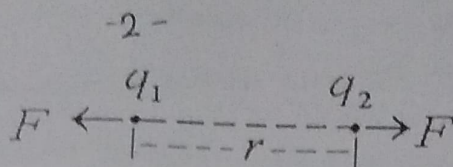
$$F \alpha \frac{q_1 q_2}{r^2} \quad \text{directly propor-}$$

inversly twm of

$$F = k \frac{q_1 q_2}{r^2} \quad p \propto k$$

Where k is constant of proportionality. In air $k = \frac{1}{4\pi\varepsilon_0}$, where $\varepsilon_0$ is the permittivity of free space. $\varepsilon_0 = 8.854 \times 10^{-12} C^2 N^{-1} m^{-2}$

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \quad F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

, Where, $\hat{r}$ is the unit vector from q1 to q2.

### 1.3. Electric field

The electric field can be defined as the force per unit charge acting at a given point. It's a vector function, and it is denoted by E. The ratio and the direction of the force will become constant as the amount of test charge is made smaller and smaller. Electric field can be written as,

$$E = \frac{F}{q}$$

Where is the force is the electric field,.q is the charge.

### 1.4 Electric field due to point charge:

Let q1 be the point charge placed at x1. A test charge q is placed at x shown in figure

$$E \leftarrow \overset{X}{\underset{q}{\bullet}} \xrightarrow{\hspace{3cm}} \overset{X_1}{\underset{q_1}{\bullet}}$$
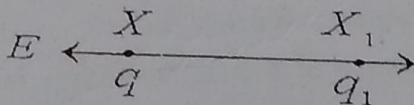
$$F = kqq_1 \cdot \frac{X - X_1}{|X - X_1|^3}$$

According to coulomb's law

The electric field at a point x is, by the definition the force per unit test charge.

$$E(X) = \frac{F}{q} = kq_1 \frac{X - X_1}{|X - X_1|^3}$$

The constant of proportionality k depends on system of unit used. In electrostatic unit (esu) k=1 and of charge is called the static coulomb and the electric field is measured in stat volt per centimeter. In S.I. system,

$$K = \frac{1}{4\pi\varepsilon_0} = 10^{-7} C^2$$

Where, $\varepsilon_0 = 8.854 \times 10^{-12}$ Farad per meter (F/m) is called the permittivity of free pace and the S.I. unit of charge is the coulomb and electric field is measured in volt per meter. In S.I. system equation (2) can be written as,

$$E(X) = \frac{1}{4\pi\varepsilon_0} q_1 \frac{X - X_1}{|X - X_1|^3} \qquad - 4$$
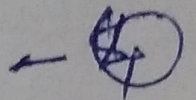
The electric field at x due to a system of point charge qi placed at xi, i=1, 2...n, then,

$$E(X) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{X - X_i}{|X - X_i|^3} \qquad - 5$$

Equation (4)

If the charges are so small and equation (5) can be described by charge density $\rho(x')$ If dq is the charge in small volume $dx' dy' dz'$ at the point $X'$ the dq= $\rho(x') dx' dy' dz'$ and the sum is replaced by an integral.

$$E(X) = \frac{1}{4\pi\varepsilon_0} \int \rho(X') \frac{X - X'}{|X - X'|^3} dx' dy' dz'$$

Substituting $dx' dy' dz'$ = d3x in equation (6) we get,

$$E(X) = \frac{1}{4\pi\varepsilon_0} \int \rho(X') \frac{X - X'}{|X - X'|^3} d^3x'$$

The law relates the flux through any closed surface and the net charge enclosed with in the surface.

## Statement:

The law states that the total flux of the electric field E over any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the net charge enclosed by the surface. Charge outside the surface will not contribute to flux.

## I roof:

Consider a point charge q and a closed surface S as shown in figure. Let r be the distance from the charge to a point on the surface n be the outwardly directed unit normal to the surface at the point, da be an element of surface area.
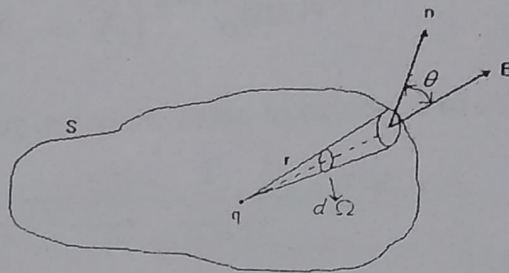


If the electric field. E at a point on the surface due to charge q makes an angle $\theta$ with the unit normal, then

$$E.n\ da = \frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}da \qquad\longrightarrow (1)$$

Substituting $\cos\theta da = r^2 d\Omega$. Where $d\Omega$ is the element of solid angle subtended by da at the position of the charge.

Therefore,
$$E.n\ da = \frac{q}{4\pi\varepsilon_0}d\Omega \qquad\longrightarrow (2)$$

If we now integrate the normal component of E over the whole surface

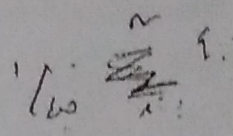$$\oint_S E.n da = \oint_S \frac{q}{4\pi\varepsilon_0}d\Omega \qquad\longrightarrow (3)$$

$$\therefore \oint d\Omega = 4\pi$$

Therefore equation (3) becomes,

$$\oint_S E.n\ da = \begin{cases} q/\varepsilon_0 \\ 0 \end{cases} \longrightarrow (4)$$

The result is Gauss's law for single point charge. For a discrete set of charges, equation (4) becomes,

$$\oint_S E.n\ da = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i \longrightarrow (5)$$

Where the sum is over only those charges inside the surface S. For continuous charge density $\rho(X)$ Gauss's law become,

$$\oint E.n\ da = \frac{1}{\varepsilon_0} \int \rho(X) d^3x \longrightarrow (6)$$

Where V is the volume enclosed by surface S.

## Gauss law application

### Field of a Charged Conducting Sphere (i)

We now know that any excess charge on a isolated solid conducting sphere is distributed uniformly on the outer surface. We also know that the field of the sphere is radial everywhere, because it similar to a point charge in that there isn't any reason why the field would deviate in any particular way from the radial direction. Again, the magnitude of the field is uniform over a spherical surface of radius r, as long as its concentric with the conducting sphere. This is shown on the illustration at right (click on the illustration for a VRML model of the sphere).

If r > R, we know that outside the sphere we can treat the sphere as if it were a point charge. Another way we can look at it is that the Gaussian surface encloses only that charge Q. And since the field lines are all perpendicular to the surface and uniform all over, we get:
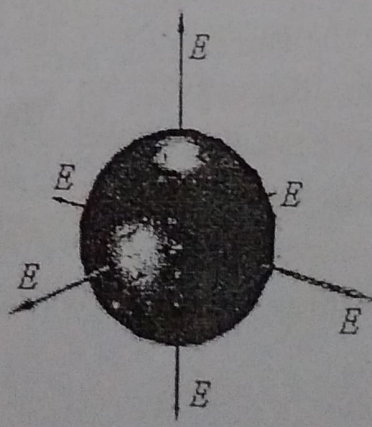
$$EA = \frac{q_{enc}}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

If r < R, the Gaussian surface would be inside the conductor. Thus, the field would be zero since all charge is on the outside surface of the conductor.

9 . Field of a Line Charge (Charged Cylindrical Conductor)

Notice that earlier with Coulomb's Law we derived the field of a thin line of charge, which we now will call simply a line charge, but now we will do it more easily with Gauss's Law. Now our wire is very long and we aren't near the ends of it, so we can say the field lines outside the wire are radial and lie in planes perpendicular to the wire (Field lines will not behave that way when you get close to the ends; they cease to be parallel to each other and that gets all messy). We also know that the field has the same magnitude at the same radial distance from the wire (Is this starting to sound familiar? It is because before we can use Gauss's Law we need to establish that the field is both constant and perpendicular to the Gaussian surface that we use.).

Knowing all that, we need to select a surface that we can use to find the field and naturally we would choose a cylinder of an radius r and length of l, with the ends of the cylinder perpendicular to the wire (parallel to the field lines). You can see the illustration at right to visualize this setup (click on the illustration for a VRML model of the wire).

$E_\perp = E$
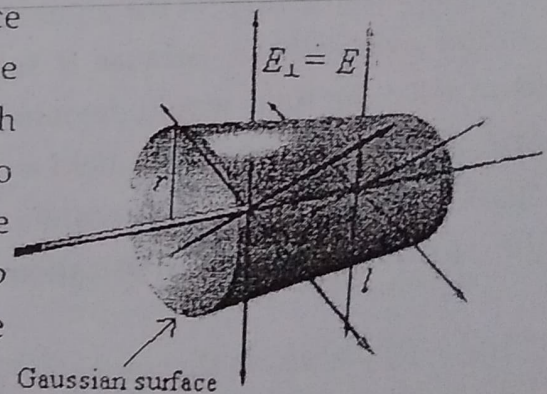
Gaussian surface

Now, □ is the charge per unit length of the wire. So that means that the charge in the Gaussian surface is □ times the length. Since E is at right angles to the wire, the component of E normal to the end faces is zero. That means that the end faces don't contribute to the integral in Gauss's law. On the other hand, the side of the cylinder is perpendicular to the field lines, so they will contribute to the integral. Since the surface area of the Gaussian surface is 2□rl>, we get this for the solution:

$$E(2\pi l) = \frac{q}{\varepsilon_0}$$

$$q = \lambda l$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Now if you notice that the difference between a charged cylindrical conductor and a line charge is thickness, you will also realize that the relationship between the cylinder and the line is the same as the sphere and the point charge. Since excess charge on a conductor collects on the surface, there is no field inside the cylinder. Outside the cylinder you notice that you can treat the cylinder like a line, and you will arrive at the same answer.

## ③ Field of an Infinite Plane Sheet of Charge

Like last time, the infinite sheet of charge has a charge per unit area of □. However, this time we are going to use Gauss's Law. Create a cylinder whose ends have an area A and whose walls are perpendicular to the sheet as shown in the illustration to the right (click on the illustration for a VRML model of the sheet). By symmetry, and the fact the sheet is infinite, the field E has the same magnitude on both sides, is uniform, and is directed normally away from the sheet. The component of E crossing the walls of the cylinder, the curved part, is zero. What that also means is:

$$\oint E_\perp dA = 2EA$$

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$
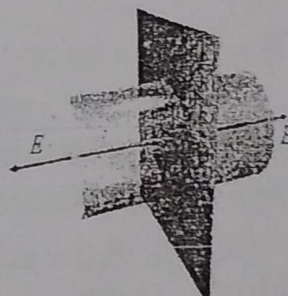


We used 2EA because we need to take into account both sides of the sheet. We see again that the magnitude of the field is independent of distance. That is because the field lines everywhere are straight, parallel, and uniformly spaced. Why this is true is because the sheet is infinitely large. Actually this is just an idealization of reality since nothing can really be infinitely large and that it can be applied to finite sheets as long as we stay away from the edges (where the field lines aren't parallel or uniformly spaced). Field of an Infinite Plane Charged Conducting Plate. If we give a flat metal plate a net charge, the charge distributes itself over the entire outer surface. This is just like two infinite plane sheet of charge, one on each side of the plate. The fields from the two sheets (named 1 and 2) will thus be superimposed on one another in each of the two directions, as you can see in the illustration at right. Thus, outside the plate, the electric field will be:

$$E = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

But what about inside? Well, the field lines of the two sides of the plate cancel each other out. The resultant of the field lines is zero, which you realize is the same as in any other conductor. Inside a conductor, there is no charge and thus no electric field. In that derivation we used a previous derivation in order to do this one. If you're a purist and want to do it the way Gauss would've wanted you to, with his law directly, it is:

$$EA = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{\varepsilon_0}$$

)Same diagram, same situation, different way. Inside the conductor, where the inside face lies, the field is zero; it is always zero inside a conductor. Outside the conductor, it is perpendicular to the plate, so the normal component of E is zero over the walls and equal to E on the outside face.

## Field between Oppositely Charged Parallel Conducting Plates

The illustration to the right shows what the field lines of two plane parallel conducting plates are. The field between the plates is basically uniformly spaced but there is a little fringing, at the edges. We are not concerned with the edges so we can assume it is uniform. These parallel plates are the basis of capacitors, which will be discussed                in                a                later                topic.
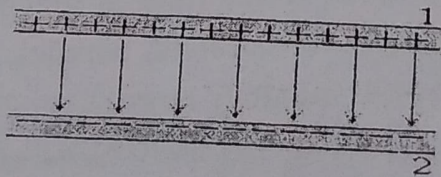


These two plates can be treated as two sheets of charge of opposite charge and we can use Gauss's Law. Outside the two sheets, the components of the electric fields are of the same magnitude but oppositely directed so their resultant is zero. At any point between the plates the components are in the same direction so their resultant is the sum. Basically giving you the same result as that of the charged conducting plate:

$$E = \frac{\sigma}{\varepsilon_0}$$

Electrical potential

In order to bring two like charges near each other work must be done. In order to separate two opposite charges, work must be done. Remember that wheneverworkgetsdone,energychangesform.

As the monkey does work on the positive charge, he increases the energy of that charge. The closer he brings it, the more electrical potential energy it has. When he releases the charge, work gets done on the charge which changes its energy from electrical potential energy to kinetic energy. Every time he brings the charge back, he does work on the charge. If he brought the charge closer to the other object, it would have more electrical potential energy. If he brought 2 or 3 charges instead of one, then he would have had to do more work so he would have created more electrical potential energy. Electrical potential energy could be measured in Joules just like any other form of energy.

Since the electrical potential energy can change depending on the amount of charge you are moving, it is helpful to describe the electrical potential energy per unit of charge. This is known as electrical potential

$$electrical\ potential = \frac{work\ -or-\ \Delta electrical\ potential\ energy}{unit\ of\ charge\ moved}$$

$$V = \frac{W}{q_{moved}}$$

As a formula it is written like this: The energy per unit of charge is often called voltage so it is symbolized with the capital letter V. Work or energy can be measured in Joules and charge is measured in Coulombs so the electrical potential can be measured in Joules per Coulomb which has been defined as a volt

$$1\frac{J}{C} = 1\ volt$$

## 1.7. Electric field and equipotential plots

We know that Coulomb's Law determines the force between two charges, and that by applying the law of superposition and using some calculus we can determine the force on a test charge from any collection of charges. From this we can abstract the concept of an electric field - an electric field is a mapping of the force that a test charge would feel due to the pre-existing configuration of charges. The equation F=qE makes this relationship clear, to get the force from the field we simply need to multiply through by q, the value of the test charge. In Physics 4A, soon after we learned about the concept of force, we also learned about the concept of work. One of the differences between force and work is that force is a vector and work is a scalar. Since one of the difficulties we experience in Physics 4B is the vector nature of the electric field, it would be helpful to find an analog to work.

In fact, we do the same trick we did to define the electric field. Whereas electric field is force per unit charge, voltage or electrical potential is defined as work per unit charge. In Physics 4A we defined work as an integral of the force over a certain path. If we were dealing with conservative forces, we discovered that the actual path did not matter, and that only the endpoints were important. In Physics 4B we'll see that the electrical potential is the integral of the electric field over a certain path, and since the electric force is conservative, we can focus only on the endpoints of the path. This will make many calculations easier to compute.

The first concept we'll want to tackle is the idea of an equipotential line. Simply put, this is path where all of the voltages are equal. We've seen maps similar to equipotential lines all of our lives, a temperature map is a good example. On the map below, the low temperatures for the day have been plotted over the United States. The people who made the map connected all of the weather stations that reported the same temperature, hence the lines on the map represent paths of equal temperature. We'll do the same for voltages and create a "weather map" for different charge distributions.
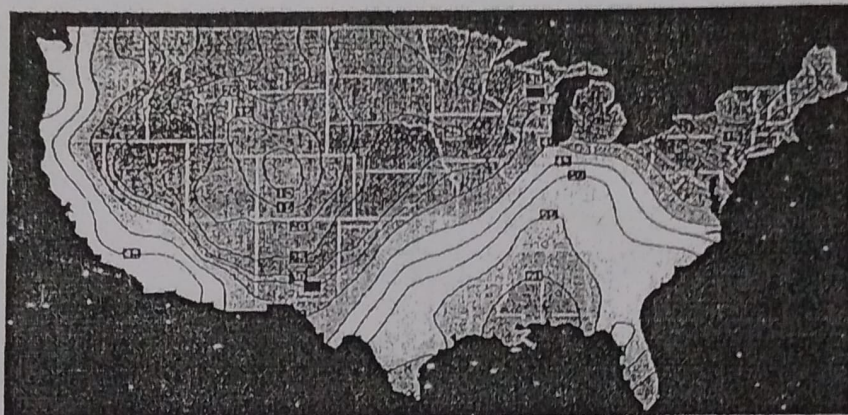


How does a map of the equipotentials help us understand the electric field (and hence the forces related to the charge distributions)? For this we need to think back to the definition of the electric potential. It is a measure of work per unit charge. If we travel along an equipotential line, since there is no change in voltage, no work is done on the test charge. Since the work done by an electric field on a charged particle is the integral of the dot product of the field and the displacement, this implies that the test charge moves at right angles to the electric field. Why? Because if the charged particle is moving at a right angle with the electric field, the dot product is zero, and hence the work is zero. Therefore equipotential lines must be at right angles to electric fields. A map of the electric field can be generated if we simply draw lines at 90 degrees from the equipotentials. (Note that since pressure and temperature variations usually match fairly well, a wind map can be approximated by drawing lines at right angles to the lines of equal temperature.)

## 1,8. Electric field in free space, conductor, dielectric

### Electric field in free space

Field strength in free space is a term in telecommunications. It is the field strength caused by a half wave dipole under ideal conditions. The actual field strength in terrestrial environments is calculated by empirical formulas based on this field strength.

N be the effective power radiated from an isotropic antenna and p be the power density at a distance d from this source

$$p = \frac{N}{4 \cdot \pi \cdot d^2}$$

Power density is also defined in terms of electrical field strength;

Let E be the electrical field and R be the impedance of the free space

$$p = \frac{E^2}{R}$$

The following relation is obtained by equating the two,

$$\frac{N}{4 \cdot \pi \cdot d^2} = \frac{E^2}{R}$$

or by rearranging the terms

$$E = \frac{\sqrt{N} \cdot \sqrt{R}}{2 \cdot \sqrt{\pi} \cdot d}$$
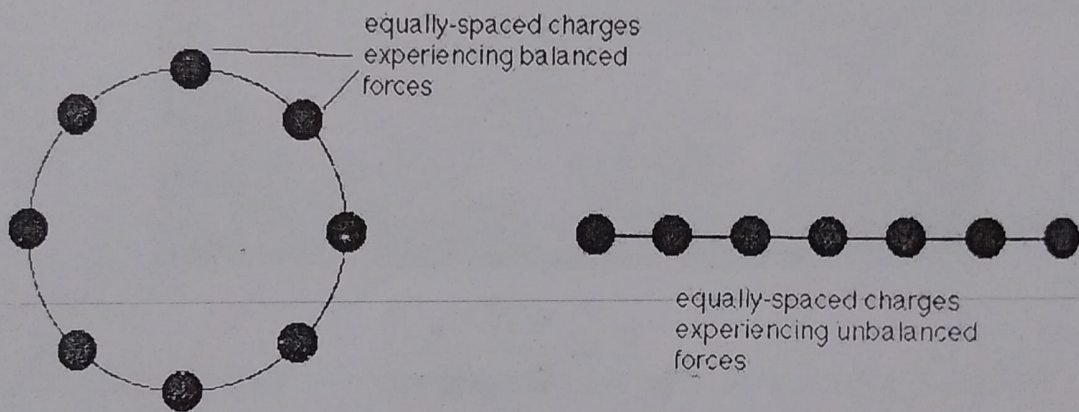
### Electric field in conductors

➢ A conductor is in electrostatic equilibrium when the charge distribution (the way the charge is distributed over the conductor) is fixed. Basically, when you charge a conductor the charge spreads itself out. At equilibrium, the charge and electric field follow these guidelines:
➢ the excess charge lies only at the surface of the conductor
➢ the electric field is zero within the solid part of the conductor
➢ the electric field at the surface of the conductor is perpendicular to the surface
➢ charge accumulates, and the field is strongest, on pointy parts of the conductor

Let's see if we can explain these things. Consider a negatively-charged conductor; in other words, a conductor with an excess of electrons. The excess

electrons repel each other, so they want to get as far away from each other as possible. To do this they move to the surface of the conductor. They also distribute themselves so the electric field inside the conductor is zero. If the field wasn't zero, any electrons that are free to move would. There are plenty of free electrons inside the conductor (they're the ones that are canceling out the positive charge from all the protons) and they don't move, so the field must be zero.

A similar argument explains why the field at the surface of the conductor is perpendicular to the surface. If it wasn't, there would be a component of the field along the surface. A charge experiencing that field would move along the surface in response to that field, which is inconsistent with the conductor being in equilibrium

Why does charge pile up at the pointy ends of a conductor? Consider two conductors, one in the shape of a circle and one in the shape of a line. Charges are distributed uniformly along both conductors. With the circular shape, each charge has no net force on it, because there is the same   amount of charge on either side of it and it is uniformly distributed. The circular conductor is in equilibrium, as far as its charge distribution is concerned.

equally-spaced charges experiencing balanced forces

equally-spaced charges experiencing unbalanced forces

With the line, on the other hand, a uniform distribution does not correspond to equilibrium. If you look at the second charge from the left on the line, for example, there is just one charge to its left and several on the right. This charge would experience a force to the left, pushing it down towards the end. For charge distributed along a line, the equilibrium distribution would look more like this:

unequally-spaced charges experiencing balanced forces
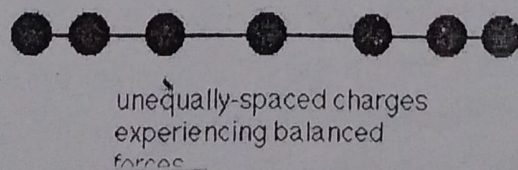
The charge accumulates at the pointy ends because that balances the forces on each charge.

Those materials which have the ability to transfer the electric effects without conducting. Dielectrics exists basically in two types
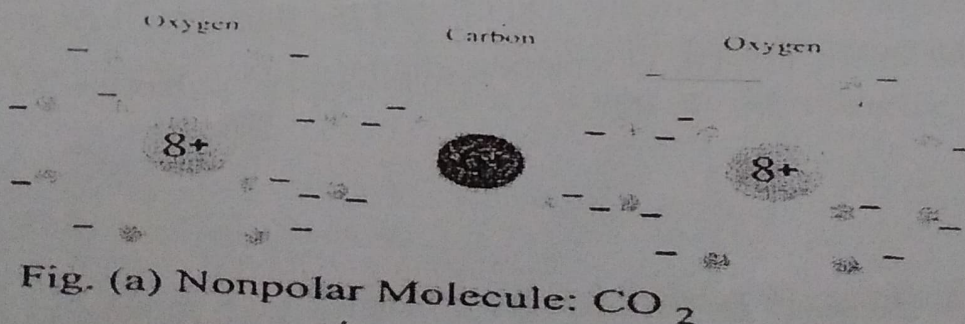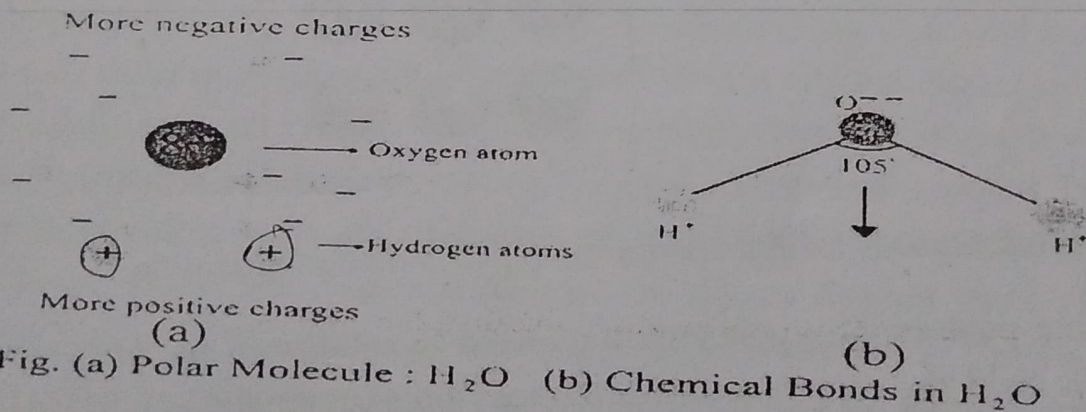
1. PolarDielectrics
2. Non polar Dielectrics

Polar Dielectrics: Polar dielectrics are those in which the possibility of center coinciding of the positive as well as negative charge is almost zero i.e. they don't coincide with each other. The reason behind this is their shape. They all are of asymmetric shape. Some of the examples of the polar dielectrics is NH3, HCL, water etc.

Non Polar dielectrics: In case of non polar dielectrics the centers of both positive as well as negative charges coincide. Dipole moment of each molecule in non polar system is zero. All those molecules which belong to this category are symmetric in nature. Examples of non polar dielectrics are: methane, benzene etc.

a)Polar molecules:

Polar means having electrical poles (i.e. electrical polarity). The molecules in which the arrangement or geometry of the atoms is such that one end of the molecule has a positive electrical charge and the other side has a negative charge are called as polar molecules. Examples of polar molecules are Water ($H_2O$) (Fig. 2.1a), Ammonia (NH3), Hydrochloric acid (HCl), Sulfur Dioxide (SO2), Hydrogen Sulfide (H2S), Carbon Monoxide (CO) etc.

More negative charges

Oxygen atom

Hydrogen atoms

More positive charges

(a)

(b)

Fig. (a) Polar Molecule : $H_2O$   (b) Chemical Bonds in $H_2O$

Oxygen            Carbon            Oxygen

8+            -13-            8+

Fig. (a) Nonpolar Molecule: $CO_2$

A non-polar molecule is that in which the electrons are distributed more symmetrically and thus does not have an excess /abundance of charges at the opposite sides. The charges all cancel out each other. e.g. $CO_2$, $H_2$, $N_2$ , $O_2$, $CH_4$, $CCl_4$ etc.

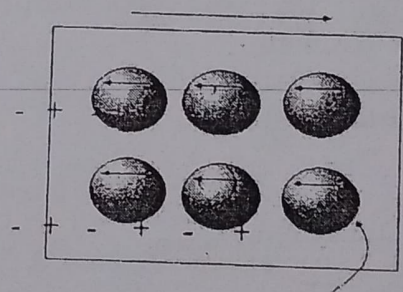## 1.9. Dielectrics Polarization

Dielectrics are materials which have no free charges; all electrons are bound and associated with the nearest atoms. An external electric field causes a small separation of the centers of the electron cloud and the positive ion core so that each infinitessimal element of volume behaves as an electric dipole. Dielectrics may be subdivided into two groups :

Non-Polarwhich behave as above Polar in which the molecules or atoms possess a permanent dipole moment which is ordinarily randomly oriented, but which become more or less oriented by

The induced dipole field opposes the applied field. In the diagram shown opposite the volume element indicated could represent an atom, a molecule, or a small region.

The type of polarization on a microscopic scale is determined by the material. Most materials exhibit polarization only in the presence of an external field. A few however show permanent polarization:



Ferroelectric crystals exhibit spontaneous permanent polarization.

Electrets become permanently polarized if allowed to solidify in the presence of a strong electric field the type of polarization may be additionally subdivided into the foll owing categories:

Electronic displacement of the electronic cloud w.r.t the nucleus. Ionic separation of +ve and -ve ions in the crystal. Orientation alignment of permanent dipoles (molecules).

Space-charge free electrons are present, but are prevented from moving by barriers such as grain boundaries - the electrons "pile up".

## 1.10. Dielectric strength

The theoretical dielectric strength of a material is an intrinsic property of the bulk material and is dependent on the configuration of the material or the electrodes with which the field is applied. The "intrinsic dielectric strength" is measured using pure materials under ideal laboratory conditions. At breakdown, the electric field frees bound electrons. If the applied electric field is sufficiently high, free electrons from background radiation may become accelerated to velocities that can liberate additional electrons during collisions with neutral atoms or molecules in a process called avalanche breakdown. Breakdown occurs quite abruptly (typically in nanoseconds), resulting in the formation of an electrically conductive path and a disruptive discharge through the material. For solid materials, a breakdown event severely degrades, or even destroys, its insulating capability.

Factors affecting apparent dielectric strength

> It decreases slightly with increased sample thickness. (see "defects" below)
> It decreases with increased operating temperature.
> It decreases with increased frequency.

for gases (e.g. nitrogen, sulfur hexafluoride) it normally decreases with increased humidity.

for air, dielectric strength increases slightly as humidity increases

### Breakdown field strength

The field strength at which breakdown occurs depends on the respective geometries of the dielectric (insulator) and the electrodes with which the electric field is applied, as well as the rate of increase at which the electric field is applied. Because dielectric materials usually contain minute defects, the practical dielectric strength will be a fraction of the intrinsic dielectric strength of an ideal, defect-free, material. Dielectric films tend to exhibit greater dielectric strength than thicker samples of the same material. For instance, the dielectric strength of silicon dioxide films of a few hundred nm to a few μm thick is approximately 0.5GV/m. However very thin layers (below, say, 100 nm) become partially conductive because of electron tunneling. Multiple layers of thin dielectric films are used where maximum practical dielectric strength is required, such as high voltage capacitors and pulse transformers. Since the dielectric strength of gases varies depending on the shape and configuration of the electrodes, it is usually measured as a traction of the dielectric strength of Nitrogen gas.

## 1.11. Electric field in multiple dielectrics:

To solve Laplace and Poisson's equation involving dielectric materials we need to derive the boundary conditions of the fields which obey at the dielectric surfaces .To find the boundary condition upon the component of a normal to the

between two different dielectrics as noted in figure.

One of the dielectric has dielectric constant $K\rho_1$, and the other has a dielectric constant $K\rho_2$. The fields in the dielectrics are D1 and D2 and their dielectrics are noted in the figure.
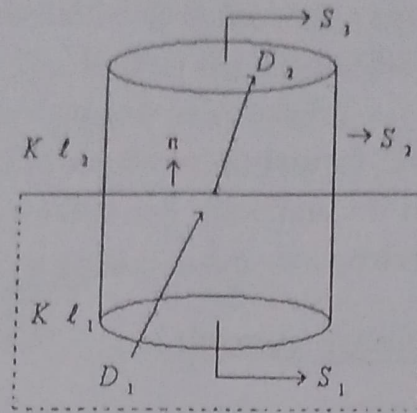


Determining the boundary conditions for D and E at a Dielectric boundary using Gauss's law:

The free charge on the surface of the interface is $\sigma$ and the vector normal to the surface $\hat{n}$ are as noted in the diagram points from the surface of the first dielectric toward the surface of the second dielectric using Gauss's law over the surface.

$$\oint D \cdot ds = Q$$ we have

$$\oint D \cdot ds = \int_{bottom} D_1 ds_1 + \int_{top} D_2 ds_2 + \int_{side} D \cdot ds_3 = Q$$

$$\longrightarrow \quad (1)$$

We shrink the pill box down so that it is so thin that any contribution due to the flux of the electric displacement through the side will be negligible. Thus we set the last term in equation(1) to be zero under this approximation our integral becomes.

$$\int_{bottom} D_1 \cdot ds_1 + \int_{top} D_2 \cdot ds_2 = Q$$

$$\longrightarrow \quad (2)$$

We also make the diameter of the pill box so small that any variation of $\sigma$ and D over the surface can be ignored and can be considered constant at the side of the pill box. Thus since $D_1$ is a constant over $S_1$ and $D_2$ is a constant over $S_2$ and since in the limit, the area of the surfaces which we denote simply as $S$ are the same we have

$$-D_1.\hat{n}s + D_2.\hat{n}s = \sigma.s \qquad\qquad \longrightarrow (3)$$

Where $\sigma$ represent the surface charge density which means that

$$(D_2 - D_1)\cdot\hat{n} = \sigma \qquad\qquad \longrightarrow (4)$$

Thus the normal component of the electric displacement vector D is discontinuous across the boundary by an amount equal to the free surface charge density lying on the interface. However if there is no free charge on the surface of the dielectric, then the normal component of D would be continuous across the boundary. For the case value $D = \varepsilon_0 keE$ we can derive the boundary conditions for the normal component of E and normal derivative of $\phi$. Thus equation (3) can be written as

$$(\varepsilon_0 ke_2 E_2 - \varepsilon_0 ke_1 E_1).\hat{n} = \sigma$$

$$(ke_2 E_2 - ke_1 E_1).\hat{n} = \frac{\sigma}{\varepsilon_0} \qquad\qquad \longrightarrow (5)$$

Which is consistent with the boundary conditions on E we derived for free space.

From $E = -\nabla\phi$ we have

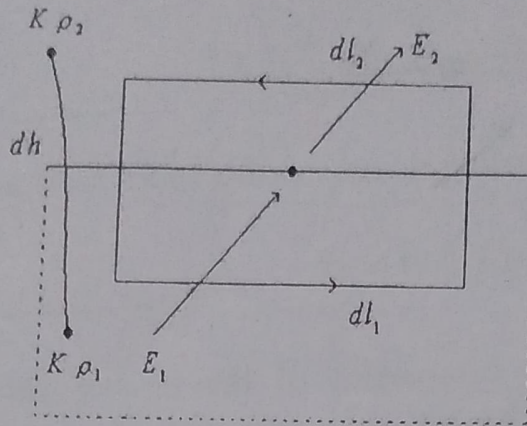$$(-ke_2 \nabla\phi_2 + ke_1 \nabla\phi_1).\hat{n} = \frac{\sigma}{\varepsilon_0} \qquad\qquad \longrightarrow (6)$$

Which can be written as?

$$ke_2 \frac{\partial\phi_2}{\partial n}\bigg|_s - ke_1 \frac{\partial\phi_1}{\partial n}\bigg|_s = \frac{-\sigma}{\varepsilon_0} \qquad\qquad \longrightarrow (7)$$

Determining the boundary conditions for D and E at a dielectric boundary using stokes's theorem:

To find the boundary condition upon the component of E to the dielectric interface we use a Gaussian pill box which straddles the boundary between two different dielectrics as noted in the figure.



One of the dielectric has dielectric constant $^{k}\rho_1$ and the other has dielectric constant $^{k}\rho_2$. The fields in the dielectrics are $E_1$ and $E_2$ and their directions are noted in figure. We use the requirement that

$$\nabla \times E = 0 \qquad\qquad\qquad\qquad (8)$$

Under all electrostatic condition which also means that

$$\oint E \cdot dl = 0 \qquad\qquad\qquad\qquad (9)$$

Using the line integral along the loop defined in figure (1) we have that

$$\int_{top} E_2 \cdot dl_2 + \int_{(1)} E \cdot dh_1 + \int_{bottom} E_1 \cdot dl_1 + \int_{(2)} E \cdot dh_2 = 0$$

$$\qquad\qquad\qquad\qquad (10)$$

Where the definition of the line segments and fields are as noted in the figure. As we shrink the height of the loop the contribution from the sides becomes negligible compared to the other integrals and can be ignored. Thus we are left with:

$$\int E_1 \cdot dl_1 + \int E_2 \cdot dl_2 = \int (E_2 - E_1) \cdot dl = 0$$

$$\qquad\qquad\qquad\qquad (11)$$

$$dl_1 = -dl_2$$

Because since our equation is true for all dl the integral is zero which gives us the result that the tangential component of E is continuous across the boundary. Thus

$$E_{1t} = E_{2t} \qquad \longrightarrow (12)$$

Which can be written in vector notation as?

$$\hat{n} \times (E_2 - E_1) = 0 \qquad \longrightarrow (13)$$

We note that the tangential component of D is not continuous across the boundary. Thus for

$$D_1 = \varepsilon_o k \rho_1 E_1 \qquad \longrightarrow (14)$$

$$D_2 = \varepsilon_o k \rho_2 E_2 \qquad \longrightarrow (15)$$

From equation (12) we have

$$\frac{D_1 t}{k\rho_1} = \frac{D_2 t}{k\rho_2} \qquad \longrightarrow (16)$$

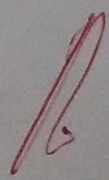The boundary conditions on $\phi$ is found by using equation

$$\int_A^B d\phi = -\int_A^B E \cdot d\ell \cdot \qquad \longrightarrow (17)$$

Which represent the relationship between the field, E and the potential $\phi$. Thus

$$\phi_2 - \phi_1 = \int (E_2 - E_1) \cdot dh \qquad \longrightarrow (18)$$

Where the line segment dh is shown in figure (1). As dh→0 the integral approaches the value zero because the field values are bounded. Thus the right side of equation (14) vanishes and we have

$$\phi_1 = \phi_2 \qquad \longrightarrow (19)$$

At the boundary. We now use this boundary condition to find the solution of boundary value problems involving dielectrics.

Summary:

Normal component of the electric displacement vector D, is discontinuous across the boundary $D_{2n} - D_{1n} = \sigma$

Tangential components of E is continuous across the boundary $E_{1t} = E_{2t}$

$$\frac{D_1 t}{k\ell_1} = \frac{D_2 t}{k\ell_2}$$

Normal component of B is continuous across the boundary $B_{1n} = B_{2n}$

Tangential component of H is discontinuous across the boundary $H_{2t} - H_{1t} = K$

### 1.12. Electric susceptibility

In electromagnetism, the electric susceptibility $X_e$ is a dimensionless proportionality constant that indicates the degree of polarization of a dielectric material in response to an applied electric field. The greater the electric susceptibility, the greater the ability of a material to polarize in response to the field, and thereby reduce the total electric field inside the material (and store energy). It is in this way that the electric susceptibility influences the electric permittivity of the material and thus influences many other phenomena in that medium, from the capacitance of capacitors to the speed of light.

Definition of volume susceptibility

Electric susceptibility is defined as the constant of proportionality (which may be a tensor) relating an electric field E to the induced dielectric polarization density P such that:

$$P = \varepsilon_0 X_e E,$$

Where $P$ is the polarization density?

$\varepsilon_0$ is the electric permittivity of free space;

$X_e$ is the electric susceptibility;

$E$ is the electric field.

The susceptibility is also related to the polarizability of individual particles in the medium by the Clausius-Mossotti relation. The susceptibility is related to its relative permittivity $\varepsilon_r$ by:

$$\chi_e = \varepsilon_r - 1 \qquad \varepsilon_r = 1 + \chi_e$$

So in the case of a vacuum:

$$\chi_e = 0$$

At the same time, the electric displacement D is related to the polarization density P by:

$$D = \varepsilon_0 E + P = \varepsilon_0(1 + \chi_e)E = \varepsilon_r \varepsilon_0 E.$$

### 1.12.2 Molecular polarizability

A similar parameter exists to relate the magnitude of the induced dipole moment p of an individual molecule to the local electric field E that induced the dipole. This parameter is the molecular polarizability and the dipole moment resulting from the local electric field Epochal is given by:

$$p = \varepsilon_0 \alpha E_{local}$$

This introduces a complication however, as locally the field can differ significantly from the overall applied field. We have:

$$P = Np = N\varepsilon_0 \alpha E_{local},$$

where P is the polarization per unit volume, and N is the number of molecules per unit volume contributing to the polarization. Thus, if the local electric field is parallel to the ambient electric field, we have:

$$\chi_e E = N\alpha E_{local}$$

Thus only if the local field equals the ambient field can we write:

$$\chi_e = N\alpha$$

### Nonlinear susceptibility

In many materials the polarizability starts to saturate at high values of electric field. This saturation can be modelled by a nonlinear susceptibility. These susceptibilities are important in nonlinear optics and lead to effects such as second harmonic generation (such as used to convert infrared light into visible light, in green laser pointers).

The standard definition of nonlinear susceptibilities in SI units is via a Taylor expansion of the polarization's reaction to electric field:

$$P = P_0 + \varepsilon_0\chi^{(1)}E + \varepsilon_0\chi^{(2)}E^2 + \varepsilon_0\chi^{(3)}E^3 + \cdots .$$

(Except in ferroelectric materials, the built-in polarization is zero, $P_0 = 0$.) The first susceptibility term, $\chi^{(1)}$, corresponds to the linear susceptibility described above. While this first term is dimensionless, the subsequent nonlinear susceptibilities $\chi^{(n)}$ have units of (m/V)n-1.

The nonlinear susceptibilities can be generalized to anisotropic materials (where each susceptibility $\chi^{(n)}$ becomes an n+1-rank tensor).

## 1.13. Electrostatic energy in dielectric media:

The energy of a system of charges in free space due to charge density $\rho(x)$ and the potential $\phi(x)$ is given by

$$W = \frac{1}{2}\int \rho(x)\phi(x)d^3x$$

$\longrightarrow$ (1)

The total work done was given by equation (1) with the dielectric media the work is done not only to bring real charge into position, but also to produce a certain state of Polarization in the medium. $\rho$ And $\phi$ in equation (1) represent macroscopic variables. Let us consider a small change in the energy $\delta W$ due to some sort of change $\delta\rho$ in the macroscopic charge density $\rho$ existing in all space. Then the work done is given by,

$$\delta W = \int \delta\rho(x)\phi(x)d^3x$$

$\longrightarrow$ (2)

Where $\phi(x)$ is the Potential due to charge density $\rho(x)$

Since $\quad \nabla.D = \rho$

We can relate the change $\delta\rho$ to a change in the displacement of $\delta D$

$$\delta\rho = \nabla.(\delta D) \longrightarrow (3)$$

Then the energy change $\delta W$ can be change into the form,

$$\delta w = \int E.\delta D d^3 x \longrightarrow (4)$$

Where $E = -\nabla\phi$ and $\rho(x)$ was a localized charge distribution? The total electrostatic energy find by Integrating between the limit D=0, D=D, then

$$W = \int d^3 x \int_0^D E.\delta D \longrightarrow (5)$$

If the medium is linear, then

$$E.\delta D = \frac{1}{2}\delta(E.D) \longrightarrow (6)$$

And the total Electrostatic Energy is

$$W = \frac{1}{2}\int E.D d^3 x \longrightarrow (7)$$

The above Electrostatic Energy equation (1) is valid macroscopically, only if the behavior is linear. Now consider the problem the change in energy when a dielectric object with a linear response is placed in an electric field whose sources are fixed Suppose that initially the electric field Eo due to certain distribution of charges $\rho_o(x)$ exist in a medium of electric susceptibility $\varepsilon_0$ which may be a function of position The initial Electrostatic energy is,

$$W_1 = \frac{1}{2}\int E_0.D_0 d^3 x \longrightarrow (8)$$

Where $D_0 = \varepsilon_o E_o$ Sources fixed in position a dielectric object of volume V1 is introduced into the field changing field from E0 to E. The presence of the object can be described by susceptibility $\varepsilon(x)$ which has value $\varepsilon_1$ inside V1 and $\varepsilon_o$ outside V1. The energy now has the value

$$W_1 = \frac{1}{2} \int E.D d^3 x \qquad \longrightarrow \quad (9)$$

Where $D = \varepsilon E$, the difference in the energy can be written as

$$W_1 = \frac{1}{2} \int (E.D_o - D.E_o) d^3 x \qquad \longrightarrow \quad (10)$$

$$= \frac{1}{2} \int (E.D - D.E_o) d^3 x + \frac{1}{2} \int (E + E_o).(D - D_o) d^3 x \qquad \longrightarrow \quad (11)$$

The second integral vanishes, since $\nabla \times (E + E_o) = 0$. So we can write

$$E + E_o = -\nabla \phi \qquad \longrightarrow \quad (12)$$

Then the second Integral becomes,

$$I = -\frac{1}{2} \int \phi \Delta.(D - D_o) d^3 x \qquad \longrightarrow \quad (13)$$

Above Integration by parts transform into,

$$I = \frac{1}{2} \int \phi \nabla.(D - D_o) d^3 x = 0 \qquad \longrightarrow \quad (14)$$

Since $\nabla.(D - D_o) = 0$, because the source charge density $\rho_o(x)$ is assumed unaltered by the insertion of the dielectric object. Consequently the energy change is,

$$W = \frac{1}{2} \int (E.D_o - D.E_o) d^3 x. \qquad \longrightarrow \quad (15)$$

The integration appears to be over all space, but is actually only over the Volume V1 of the object, since, outside V1, $D = \varepsilon_o E$. Therefore, we can write

$$W = -\frac{1}{2} \int_{V_1} (\varepsilon_1 - \varepsilon_0) E.E_0 d^3 x \qquad \longrightarrow (16)$$

If the medium surrounding the dielectric body is free space, then using the definition of Polarization P, equation (16) can be expressed in the form

$$W = -\frac{1}{2}\int_{V_1} P.E_0 d^3x. \tag{17}$$

Where P is the Polarization of the dielectric. This shows that the energy density of a dielectric placed in a field $E_0$ whose source are fixed is given by

$$W = -\frac{1}{2}P.E_o \tag{18}$$

This result is analogous to the dipole term in the energy of a charge distribution in an external field. The factor $\frac{1}{2}$ is due to the fact that equation (18) represents the energy density of a Polarizable dielectric in an external field, rather than a permanent dipole. It is the same factor $\frac{1}{2}$ that appears in (6).Equation (16) and (17) show that a dielectric body will tend to move toward regions of increasing field Eo provided $\varepsilon_1 > \varepsilon_o$. To calculate the force acting we can imagine a small generalized displacement of the body $\delta\xi$. Then there will be a change in the energy $\delta w$. Since the charges are held fixed, there is no external source of energy and the change in field energy can be interpreted as a change in the potential energy of the body. This means that there is force acting on the body.

$$F_\xi = -\left(\frac{\partial w}{\partial \xi}\right)_Q \tag{19}$$

Where the subscript Q has been placed on the partial derivative to indicate that the sources of the field are kept fixed. In practical situations involving the motion of dielectrics the electric fields are often produced by a configuration of electrodes held at fixed potentials by connection to an external source such as a battery. To maintain the potentials constant as the distribution of dielectric varies, charge will flow to or from the battery to the electrodes. This means that the energy is being supplied from the external source, and it is of interest to compare the energy supplied in that way with the energy change found above for fixed source of the field we will treat only linear media so that (1) is valid. It is sufficient to consider small changes in an

existing configuration. From (1) it is evident that the change in energy accompanying the changes $\delta\rho(x)$ and $\delta\Phi(x)$ in charge density and potential is

$$\delta W = \frac{1}{2} \int (\rho\delta\Phi + \Phi\delta\rho) d^3x \qquad\qquad (20)$$

Comparison with (2) shows that, if the dielectric Properties are not changed the two term in (20) are equal. If, however, the dielectric properties are altered,

$$\varepsilon(x) \rightarrow \varepsilon(x) + \delta\varepsilon(x) \qquad\qquad (21)$$

The contributions in (20) are not necessarily the same. Intact, we have just calculated the change in energy brought about by introducing a dielectric body into an electric field whose source was fixed $(\delta\rho = 0)$. Equal contributions in (20) would imply $\delta W = 0$, but (15) or (16) are not zero in general. The reason for this difference lies in the existence of the polarization charge. The change in dielectric properties implied by (21) can be thought of as a change in the polarization - charge density. If then (20) is interpreted as an integral over both free and polarization - charge densities (i.e. a microscopic equation) the two contribution are always equal. However it is often convenient to deal with macroscopic quantities. Then the equality holds only if the dielectric properties are unchanged.

The process of altering the dielectric properties in some way in the presence of electrodes at fixed potentials can be viewed as taking placed in two steps. In the first step the electrodes are disconnected from the batteries and the charges on them held fixed $(\delta\rho = 0)$. With the change (21) in dielectric properties, the energy change is

$$\delta W_1 = \frac{1}{2} \int \rho\delta\Phi_1 d^3x \qquad\qquad (22)$$

Where $\delta\Phi_1$ is change in potential produced? This can be shown to yield the result (16). If the second step the batteries are connected again to the electrodes to restore their potentials to the original values. There will be a flow of charge $\delta\rho_2$ from the batteries accompanying the change in potential $\delta\Phi_1 = -\delta\Phi_1$. Therefore the energy change in the second step is

$$\delta W_2 = \frac{1}{2}\int(\rho\delta\Phi_2 + \Phi\delta\rho_2)d^3_x = -2\delta W_1$$

$$\longrightarrow \quad (23)$$

Since the two contributions are equal. In the second step we find the external source changing the energy in the opposite sense and by twice the amount the initial step the net change is

$$\delta W = -\frac{1}{2}\int \rho\delta\Phi_1 d^3x$$

$$\relbar\joinrel\longrightarrow \quad (24)$$

Symbolically,

$$\delta W_V = -\delta W_Q$$

$$\longrightarrow \quad (25)$$

Where the subscript denotes the quantity held fixed. If a dielectric with $\varepsilon/\varepsilon_0 > 1$ moves into a region of greater field strength, the energy increases instead of decreases.

## 1.14 Clausius-Mossoti equation:

Let us now investigate what a dielectric equation of state actually looks like. Suppose that a dielectric medium is made up of identical molecules which develop a dipole moment

$$p = a\varepsilon_0 E$$

$$p = \alpha\varepsilon_0 E$$
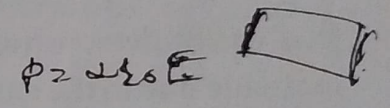
When placed in an electric field $E$. The constant $a$ is called the molecular polarizability. If $N$ is the number density of such molecules then the polarization of

$$P = \frac{N_A \rho_m a}{M}\varepsilon_0 E,$$

the medium is

$$P = Np = N a \varepsilon_0 E,$$

where $\rho_m$ is the mass density, $N_A$ is Avogadro's number, and $M$ is the molecular weight. But, how does the electric field experienced by an individual molecule relate to the average electric field in the medium? This is not a trivial question since we

$$\sigma_{pol} = -P \cos \theta$$

$$E_s = \frac{P}{2\epsilon_0} \int_0^\pi \cos^2 \theta \sin \theta \, d\theta = \frac{P}{3\epsilon_0}.$$

expect the electric field to vary strongly (on atomic length-scales) inside the dielectric.

Suppose that the dielectric is polarized with a mean electric field $E_0$ which is uniform (on macroscopic length-scales) and directed along the $z$-axis. Consider one of the molecules which constitute the dielectric. Let us draw a sphere of radius $a$ about this particular molecule. This is intended to represent the boundary between the microscopic and the macroscopic range of phenomena affecting the molecule. We shall treat the dielectric outside the sphere as a continuous medium and the dielectric inside the sphere as a collection of polarized molecules. According to Eq. (3.29) there is a polarization surface charge of magnitude

on the inside of the sphere, where $(r, \theta, \varphi)$ are spherical polar coordinates, and $P = P\hat{z} = \epsilon_0(\epsilon - 1)E_0 \hat{z}$ is the uniform polarization of the dielectric. The magnitude of $E_s$ at the molecule due to the surface charge is

$$E_s = -\frac{1}{4\pi\epsilon_0} \int \frac{\sigma_{pol} \cos \theta}{a^2} \, dS,$$

Where $dS = 2\pi a^2 \sin \theta \, d\theta$ is a surface element of the sphere. It follows that

It is easily demonstrated that $E_\theta = E_\varphi = 0$ at the molecule. Thus, the field at the molecule due to the surface charges on the sphere is

$$E = \frac{P}{3\epsilon_0}.$$

The field due to the individual molecules within the sphere is obtained by summing over the dipole fields of these molecules. The electric field at a distance $r$ from a dipole $p$ is

$$E = -\frac{1}{4\pi\epsilon_0} \left[ \frac{p}{r^3} - \frac{3(p \cdot r)r}{r^5} \right].$$

It is assumed that the dipole moment of each molecule within the sphere is the same, and also that the molecules are evenly distributed throughout the sphere. This being the case, the value of $E_s$ at the molecule due to all of the other molecules within in the sphere,

$$E_x = -\frac{1}{4\pi\,\epsilon_0}\sum_{mols}\left[\frac{p_x}{r^3} - \frac{3(p_x\,xz + p_y\,yz + p_z\,z^2)}{r^5}\right],$$

is zero, since

$$\sum_{mols} x^2 = \sum_{mols} y^2 = \sum_{mols} z^2 = \frac{1}{3}\sum_{mols} r^2$$

$$\sum_{mols} xy = \sum_{mols} yz = \sum_{mols} xz = 0.$$

It is easily seen that $E_\theta = E_\varphi = 0$. Hence, the electric field at the molecule due to the other molecules within the sphere vanishes.

It is clear that the net electric field seen by an individual molecule is

$$E = E_0 + \frac{P}{3\epsilon_0}.$$

This is larger than the average electric field $E_0$ in the dielectric. The above analysis indicates that this effect is ascribable to the long range (rather than the short range) interactions of the molecule with the other molecules in the medium. Making use of Eq. (3.88) and the definition $P = \epsilon_0(\epsilon - 1)E_0$, we obtain

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{N_A\,\rho_m\alpha}{3M}.$$

This is called the Clausius-Mossotti relation. This formula is found to work pretty well for a wide class of dielectric liquids and gases. The Clausius-Mossotti relation yields ⏋

## 1.15. Poisson's Equation

Poisson's Equation is a differential equation which is obtained for space distribution of charge on the basis of Gauss law. And the equation is given by,

$$div.grad\ v = -\frac{\rho}{\varepsilon_0}$$

$$(i.e)\nabla.\nabla v = \nabla^2 v$$

$$(or)\nabla^2 v = -\frac{\rho}{\varepsilon_0}$$

## Laplace Equation:

According to definition, Laplace Equation is given by,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$(i.e)div.grad\ v = 0\ (or)\ \nabla^2 v = 0$$

### Poisson's And Laplace Equation:

The static electric field E is described by the equation,

$$\nabla.E = 4\pi\rho \quad\longrightarrow\quad (1)$$

$$\nabla \times E = 0 \quad\longrightarrow\quad (2)$$

$$E = -\nabla\phi \quad\longrightarrow\quad (3)$$

Where E is gradient of scalar function, $\phi$ is Scalar potential.

Substitute the value of E (equation (3)) in equation (1)

$$\nabla.(-\nabla\phi) = 4\pi\rho$$

$$-\nabla^2\phi = 4\pi\rho$$

$$(or)\nabla^2\phi = -4\pi\rho \quad\longrightarrow\quad (4)$$

This equation is called the Poisson's Equation.

If the medium under consideration contain no free charges then equation (4) reduces to Laplace equation (i.e.) in free space $\rho = 0$, then the scalar function $\phi$ satisfy Laplace equation.

$$\therefore \nabla^2 \phi = 0 \xrightarrow{\hspace{5cm}} (5)$$

This is called Laplace's equation.

### 1.16. Boundary value problems:

Many problems in electrostatics involve boundary surfaces on which either the potential or the surface-charge density is specified. The formal solution of such problems was presented using the method of Green functions. In practical situations (or even rather idealized approximations to practical situations) the discovery of the correct Green function is sometimes easy and sometimes not. Consequently a number of approaches to electrostatic boundary-value problems have been developed, some of which are only remotely connected to the Green function method. One of the methods are the method of images, which is closely related to the use of Green functions. We shall now discuss a general method for solving electrostatic problems, without specifically solving a differential equation. The method of images concerns itself with the problem of one or more point charges in the presence of boundary surfaces. For example, conductors either grounded or held at fixed potentials. A small number of suitable placed charges of appropriate magnitudes, external to the region of interest, can simulate the required boundary conditions. These charges are called image charges and the replacement of the actual problem with boundaries by an enlarged region with image charges but not boundaries is called the method of images.

A simple example is a point charge located in front of an infinite plane conductor at zero potential as shown in Fig A. It is clear that this is equivalent to the problem of the original charge and an equal and opposite charge located at the mirror image point behind the plane defined by the position of the conductor.
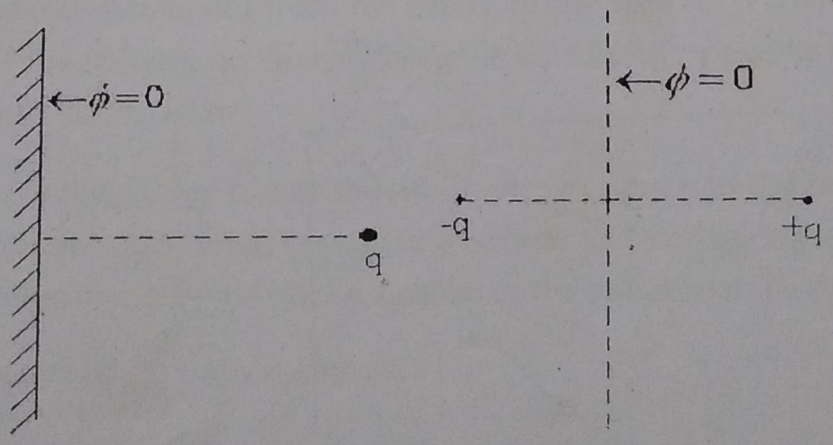


Fig.1 Solution by method of images. The original potential problem is on the left, the equivalent image problem on the right.

## Point Charge In The Presence Of a Grounded Conducting Sphere:

Consider a conducting sphere of radius a maintained at zero potential and a charge q at a distance y from the centre of the sphere. Suppose the centre of the sphere coincides with the origin. Since charge q is outside the sphere, the image position y' will lie inside
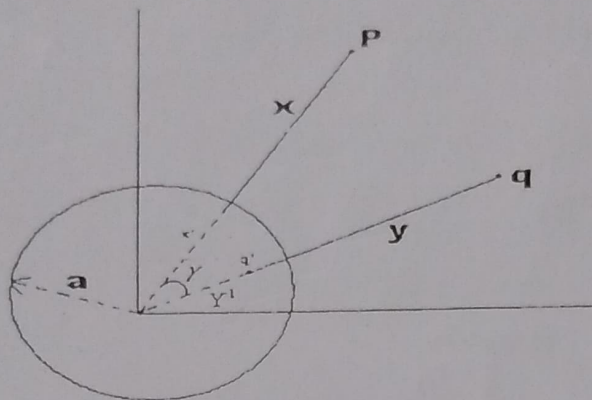


Conducting sphere of radius a with charge and image charge $q'$. The potential due to the charges $q$ and $q'$ is,

$$\phi(x) = \frac{q / 4\pi\varepsilon_0}{|x - y|} + \frac{q' / 4\pi\varepsilon_0}{|x - y'|} \longrightarrow (1)$$

We must now try to choose $q'$ and $|y'|$ such that this potential vanishes at $|x|$ = a.

If n is a unit vector in the direction x, and $n'$ a unit vector in the direction y, then

$$\phi(x) = \frac{q / 4\pi\varepsilon_0}{|xn - yn'|} + \frac{q' / 4\pi\varepsilon_0}{|xn - y'n'|} \longrightarrow (2)$$

If x is factored out of the first term and $y'$ out of the second the potential at $x = a$

$$\phi(x = a) = \frac{q/4\pi\varepsilon_0}{|an - yn'|} + \frac{q'/4\pi\varepsilon_0}{|an - y'n'|}$$

becomes

$$= \frac{q/4\pi\varepsilon_0}{a\left|n - \frac{y}{a}n'\right|} + \frac{q'/4\pi\varepsilon_0}{y'\left|n' - \frac{a}{y'}n\right|} \longrightarrow (3)$$

From the equation (3) it will be seen that the choices:

$$\frac{q}{a} = \frac{-q'}{y'}$$

$$\frac{y}{a} = \frac{a}{y'}$$

Make $\phi(x = a) = 0$, for all possible values of $n.n'$. Hence the magnitude and position of the image charge are

$$q' = \frac{-a}{y}q; \qquad y' = \frac{a^2}{y} \longrightarrow (4)$$

We note that as the charge q is brought closer to the sphere, the image charge grows in magnitude and moves out from the centre of the sphere. Where q is just outside the surface of the sphere, the image charge is equal and opposite in magnitude and lies just beneath the surface.

Now that the image charge has been found, we can return to the original problem of a charge q outside a grounded conducting sphere and consider various effects. The actual charge density induced on the surface of the sphere can be calculated from the normal derivative of $\phi$ at the surface.

$$\frac{1}{\left|xn - yn'\right|}\Bigg|_{x=a} = \frac{1}{\left(x^2 + y^2 - 2xy\cos\gamma\right)^{1/2}}\Bigg|_{x=a} = \frac{y}{y}\frac{1}{\left(x^2 + y^2 - 2xy\cos\gamma\right)^{1/2}}\Bigg|_{x=a}$$

$$= \frac{1}{y}\left[\frac{1}{\left(\dfrac{x^2}{y^2} + 1 - 2\dfrac{x}{y}\cos\gamma\right)^{1/2}}\Bigg|_{x=a}\right] = \frac{1}{y}\left(1 + x^2\Big/y^2 - 2\frac{x}{y}\cos\gamma\right)^{-1/2}\Bigg|_{x=a}$$

$$\longrightarrow \tag{5}$$

$$\left[\because y' = \frac{a^2}{y}\right]$$

$$= \left[x^2 + \frac{a^4}{y^2} - 2x\frac{a^2}{y}\cos\gamma\right]^{-1/2}\Bigg|_{x=a} \longrightarrow \tag{6}$$

Substituting equation (5) and (6) in (2)

$$\phi(x = a) = \frac{q}{4\pi\varepsilon_0}\frac{1}{y}\left[1 + \frac{x^2}{y^2} - 2\frac{x}{y}\cos\gamma\right]^{-1/2}\Bigg|_{x=a}$$

$$- \frac{q}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\left[x^2 + \frac{a^4}{y^2} - 2x\frac{a^2}{y^2}\cos\gamma\right]^{-1/2}\Bigg|_{x=a} \qquad \left[q' = \frac{-a}{u}q\right]$$

Differentiating with respect to $x$,

$$-\frac{\partial \phi}{\partial x}\bigg|_{x=a} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{y}\right)\left(\frac{-1}{2}\right)\left(1+\frac{x^2}{y^2}-\frac{2x}{y}\cos\gamma\right)^{-1/2-1}_{x=a}\left(\frac{2x}{y^2}-\frac{2\cos\gamma}{y}\right)_{x=a}$$

$$-\frac{q}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\left(\frac{-1}{2}\right)\left(x^2+\frac{a^4}{y^2}-2x\frac{a^2}{y}\cos\gamma\right)^{-1/2-1}_{x=a}\left(2x-\frac{2a^2\cos\gamma}{y}\right)_{x=a}$$

$$=\frac{-q}{4\pi\varepsilon_0}\frac{\cancel{2}}{\cancel{2}.y}\left(1+\frac{x^2}{y^2}-\frac{2x}{y}\cos\gamma\right)^{-3/2}\left(\frac{x}{y^2}-\frac{\cos\gamma}{y}\right)$$

$$+\frac{q}{4\pi\varepsilon_0}\frac{\cancel{2}a}{\cancel{2}y}\left(x^2+\frac{a^4}{y^2}-2x\frac{a^2}{y}\cos\gamma\right)^{-3/2}\left(x-\frac{a^2\cos\gamma}{y}\right),$$

$$=\frac{-q}{4\pi\varepsilon_0}\frac{1}{y}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{-3/2}\left(\frac{a}{y^2}-\frac{\cos\gamma}{y}\right)$$

$$+\frac{q}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\left(a^2+\frac{a^4}{y^2}-2a\frac{a^2}{y}\cos\gamma\right)^{-3/2}\left(a-\frac{a^2\cos\gamma}{y}\right)$$

$$=\frac{-q}{4\pi\varepsilon_0}\cdot\frac{1}{a.y}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{-3/2}\left(\frac{a^2}{y^2}-\frac{a\cos\gamma}{y}\right)$$

$$+ \frac{q}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\frac{\left(a - \dfrac{a^2\cos\gamma}{y}\right)}{\left(a^2 + \dfrac{a^4}{y^2} - \dfrac{2a^3}{y}\cos\gamma\right)^{-3/2}}$$

$$= \frac{-q}{4\pi\varepsilon_0}\cdot\frac{1}{a.y}\frac{\left(\dfrac{a^2}{y^2} - \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$+ \frac{q}{4\pi\varepsilon_0}\left(\frac{a^2}{y}\right)\frac{\left(1 - \dfrac{a\cos\gamma}{y}\right)}{(a^2)^{\frac{3}{2}}\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{a.y}\frac{\left(\dfrac{a^2}{y^2} - \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$+ \frac{q}{4\pi\varepsilon_0}\cdot\frac{a^2}{a^3 y}\frac{\left(1 - \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$
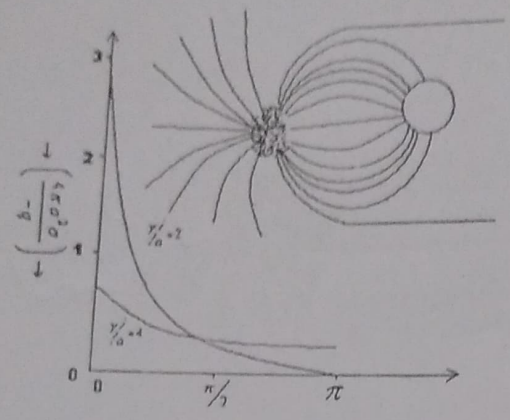
$$\left.\frac{\partial\phi}{\partial x}\right|_{x=a} = \frac{-q}{4\pi\varepsilon_0}\frac{1}{a.y}\frac{\left(\dfrac{a^2}{y^2} - \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$+ \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{a.y} \frac{\left(1 - \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{a.y} \frac{\left(1 - \dfrac{a\cos\gamma}{y} - \dfrac{a^2}{y^2} + \dfrac{a\cos\gamma}{y}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0 a^2} \left(\frac{a}{y}\right) \frac{\left(1 - \dfrac{a^2}{y^2}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

Susrface Charge Density

$$\sigma = -\varepsilon_0 \left.\frac{\partial\phi}{\partial x}\right|_{x=a}$$

$$= -\varepsilon_0 \cdot \frac{q}{4\pi\varepsilon_0 a^2} \left(\frac{a}{y}\right) \frac{\left(1 - \dfrac{a^2}{y^2}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$\sigma = -\frac{q}{4\pi a^2} \left(\frac{a}{y}\right) \frac{\left(1 - \dfrac{a^2}{y^2}\right)}{\left(1 + \dfrac{a^2}{y^2} - \dfrac{2a}{y}\cos\gamma\right)^{3/2}}$$

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Where $\gamma$ is angle between x and y. This charge density in units of $-q/4\pi a^2$ is shown plotted in Fig as a function of $\gamma$ for two values of $y/a$.



Surface charge density $\sigma$ induced on the grounded sphere of radius a as a result of the presence of a point charge q located at distance y away from the centre of the sphere, $\sigma$ is plotted in units of $\left(-q/4\pi a^2\right)$ as a function of the angular position $\gamma$ away from the radius to the charge for y=2a,4a. Inset shows lines of force for y=2a. The concentration of charge in the direction of the point charge q is evident, especially for y/a=2, it is easy to show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge, as it must be according to Gauss's law. The force acting on the charge q can be calculated in different ways. One way (easiest) is to write down immediately the force between the charge q and the image charge $q'$ The distance between them is

$$y - y' = y\left(1 - \frac{a^2}{y^2}\right)$$

Hence the attractive force, according to Coulomb's law is

$$|F| = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2}$$

For large separation the force is an inverse cube law, but close to the sphere it is proportional to the inverse square of the distance away from the surface of the sphere. The alternative method for obtaining the force is to calculate the total force acting on the surface of the sphere. The force on each element of area da is $\left(\sigma^2/2\varepsilon_0\right)$ da, where $\sigma$ is given by equation (5) as indicated in Fig 2.5.

$$d\vec{F} = \left(\sigma^2/2\varepsilon_0\right) da$$



$da$
$\gamma$
$q$
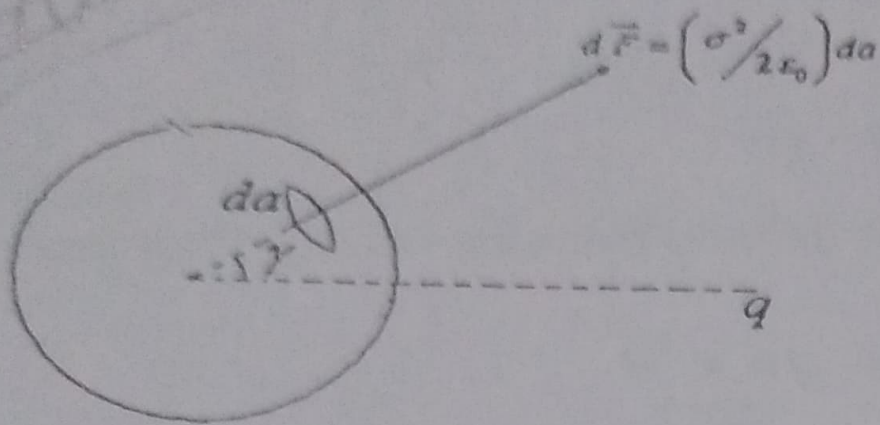
The total force acting on the sphere is given by the integral,

$$|F| = \frac{q^2}{32\,\pi\varepsilon_0 a^2}\left(\frac{a}{y}\right)^2\left(1 - \frac{a^2}{y^2}\right)^2 \int \frac{\cos\gamma}{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)^3}\, d\Omega$$

The whole discussion has been based on the understanding that the point charge $q$ is outside the sphere. Actually, the results apply equally for the charge $q$ inside the sphere.

end