☐ EULER'S METHOD

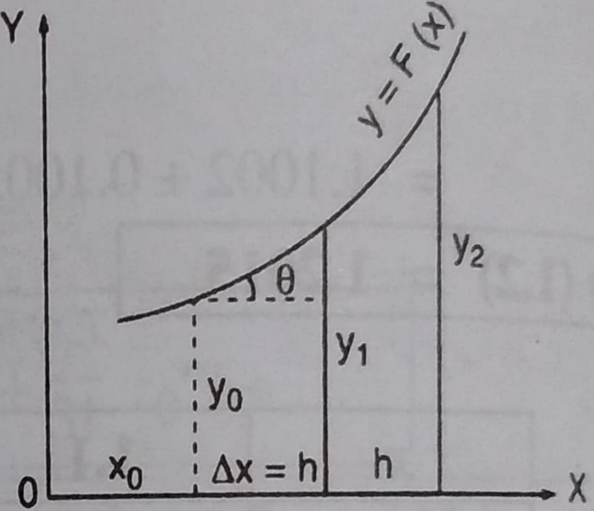
Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \ldots (1)$$

Let us solve this differential equation under the condition $y(x_0) = y_0$. The solution of (1) gives y as a function x, which may be written symbolically as

$$y = F(x) \quad \ldots (2)$$

The graph of (2) is a curve in the XY plane; and since a smooth curve is practically straight for a short distance from any point on it, we have from figure



$$\tan\theta \approx \frac{\Delta y}{\Delta x}$$

i.e.,

$$\Delta y \approx \Delta x \tan\theta$$

$$= \Delta x \left(\frac{dy}{dx}\right)_0 \quad \ldots (3)$$

$$\left[\because \text{Slope at } (x_0, y_0) = \left(\frac{dy}{dx}\right)_0 = \tan\theta\right]$$

$$\therefore y_1 = y_0 + \Delta y$$

$$\therefore y_1 = y_0 + \left(\frac{dy}{dx}\right)_0 \Delta x$$

$$y_1 \approx y_0 + f(x_0, y_0)\, h \quad [\text{Assuming } \Delta x = h]$$

$$\left[\because \frac{dy}{dx} = f(x, y) \text{ from (1)} \right.$$

$$\left. \therefore \left(\frac{dy}{dx}\right)_0 = f(x_0, y_0)\right]$$

The next value of y corresponding to $x = x_2 (= x_1 + h)$ is

$$y_2 \approx y_1 + \left(\frac{dy}{dx}\right)_1 h$$

i.e.,

$$y_2 \approx y_1 + f(x_1, y_1)\, h \quad \left[\because \left(\frac{dy}{dx}\right)_1 = f(x_1, y_1)\right]$$

$$y_3 \approx y_2 + f(x_2, y_2)\, h, \text{ etc.}$$

In general $\boxed{y_{n+1} \approx y_n + f(x_n, y_n)\, h}$

By taking h small enough and proceeding in this manner, we could tabulate the expression (2) as a set of corresponding values of x and y. **This is method is given by Euler.** This method is either too slow (in case of h is small) or too inaccurate (in case h is not small) for practical use.

❑ IMPROVED EULER'S METHOD

Let the given first order differential equation be

$$\frac{dy}{dx} = f(x, y) \qquad \ldots (1)$$

Let us solve this equation under the condition $y(x_0) = y_0$.

Starting with the initial value y_0, an approximate value for y_1 is computed from the relation

$$y_1 \approx y_0 + f(x_0, y_0)\, h \qquad \ldots (2)$$

Substituting this approximate value of y_1 in (1) we get an approximate value of $\frac{dy}{dx}$ at (x_1, y_1)

i.e.,

$$\left(\frac{dy}{dx}\right)_1^{(1)} = f[x_1, y_1^{(1)}]$$

Now an improved value of Δy, is found by multiplying h with the mean values of $\frac{dy}{dx}$ at x_0 and x_1.

i.e.,

$$\Delta y = \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2} \cdot h$$

$$= \left\{ \frac{f(x_0, y_0) + f[x_1, y_1^{(1)}]}{2} \right\} h$$

$$= \frac{h}{2} \{ f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)] \}$$

Now $y_1 = y_0 + \Delta y$

$$= y_0 + \frac{h}{2} \{ f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)] \}$$

In general

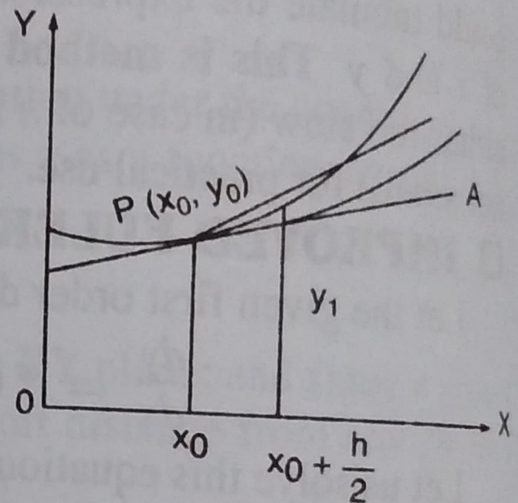
$$y_{m+1} = y_m + \frac{h}{2} \{ f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)] \}$$

This formula is called **Improved Euler's Formula.**

❑ Modified Euler's Method

In Improved Euler's method the solution curve is approximated in the interval $[x_0, x_0 + h]$ by a straight line. This line is passing through (x_0, y_0) whose slope is the averages of the slopes *viz.*,

$$\frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^1}{2}.$$



But in modified Euler's method the curve is approximated by averaging the points.

Let P (x_0, y_0) be the point on the solution curve. Let PA be the tangent at (x_0, y_0) to the curve. Let this tangent meet the ordinate at Q $\left(x_0 + \frac{h}{2}\right)$ at P_1. The Y-co-ordinate of the point P_1 is $y_0 + \Delta y$ ... (1) where Δy is the small increment along QP_1.

Now considering the triangle PP_1M, we have $\tan \theta = \frac{\Delta y}{\Delta x}$.

$$\text{i.e., } \Delta y = (\tan \theta) \Delta x = \left(\frac{dy}{dx}\right) \cdot \frac{h}{2}$$

$$\Delta y = \frac{h}{2} f(x_0, y_0) \qquad \text{... (2)}$$

[Here $\Delta x = \frac{h}{2}$]

Substituting (2) in (1) we get the Y- co-ordinate of the point P_1 is $y_0 + \frac{h}{2} f(x_0, y_0)$

Hence the co-ordinates of the point P_1 is

$$\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right] \quad \ldots (3)$$

The slope at P_1 is $\left(\frac{dy}{dx}\right)$ at P_1.

$$\text{But} \left(\frac{dy}{dx}\right) = f(x, y) \text{ (Given diff-equation)}$$

$$\therefore \left(\frac{dy}{dx}\right) \text{ at } P_1 = f\left\{ x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right\} \quad \ldots (4)$$

[∵ Replacing in (4) : x by $x_0 + \frac{h}{2}$ and y by $y_0 + \frac{h}{2} f(x_0, y_0)$]

Now draw a line P_1B with this slope (slope at P_1). Then draw a line through P (x_0, y_0) and parallel to the line P_1B. This line is taken to be the approximation to the curve in the interval $(x_0, x_0 + h)$. The equation of this line viz., PC is

$$y - y_0 = f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right](x - x_0) \quad \ldots (5)$$

[using the formula $y - y_0 = m(x - x_0)$ which is the equation of the straight line passing through (x_0, y_0) and having slope m]

Let this line (5) meets the ordinate $x = x_1 = (x_0 + h)$ at the point (x_1, y_1). Since (x_1, y_1) lies on the line (5), we have

$$y_1 - y_0 = (x_1 - x_0) f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

$$= h f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

$$\text{i.e., } y_1 = y_0 + h f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

In general,

	$y_{n+1} = y_n + h f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right]$
(or)	$y(x + h) = y(x) + h f\left[x + \frac{h}{2}, y + \frac{h}{2} f(x, y) \right]$

This formula is called **modified Euler's formula.**

Note : 1. **The error in the solution can be reduced by decreasing the step size.**

2. **The method will provide error-free solutions if the given function is linear.**

3. **Because in Euler's method we use straight-line segments to approximate the solution, this method is referred to as a first order method.**

4. **The error in one-step of the modified Euler method is $0(h^3)$. This is called the local error. There is an accumulation of error from step to step, so that the error over the whole range is $0(h^2)$ and is called global error.**

Example 1

Using Improved Euler's method find y (0.2) and y (0.4) from $y' = x + y, y(0) = 1$ with $h = 0.2$. *[A.U May 2000]*

Solution

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \ldots (1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad \ldots (2)$$

Here $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + y,\ h = 0.2$

$$\therefore f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1 \quad \ldots (3)$$

Substituting (3) in (2) we get

$$y_1 = y_0 + \frac{h}{2}\{1 + f[x_0 + h, y_0 + h \cdot 1]\}$$

$$= 1 + 0.1\{1 + f[x_0 + h, y_0 + h]\}$$

$$= 1 + 0.1\{1 + f(0 + 0.2, 1 + 0.2)\}$$

$$= 1 + 0.1\{1 + f(0.2, 1.2)\} \quad \ldots (4)$$

Now $f(0.2, 1.2) = 0.2 + 1.2 = 1.4 \quad \ldots (5)$

Substituting (5) in (4) we get

$$y_1 = 1 + 0.1\{1 + 1.4\}$$
$$= 1 + (0.1)(2.4) = 1 + 0.24$$
$$y_1 = 1.24$$

$$\therefore \; y(0.2) = 1.24$$

Putting $m = 1$ in (1) we get

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + h f(x_1, y_1)]\} \quad \ldots (6)$$

Here $x_1 = 0.2, y_1 = 1.24, \; h = 0.2$

Now $f(x_1, y_1) = f(0.2, 1.24) = 0.2 + 1.24$

$$f(x_1, y_1) = 1.44 \quad \ldots (7)$$

Substituting (7) in (6) we get

$$y_2 = 1.24 + \frac{0.2}{2}\{1.44 + f[0.2 + 0.2 \; , 1.24 + 0.2(1.44)]\}$$
$$y_2 = 1.24 + 0.1\{1.44 + f(0.4, 1.528)\} \quad \ldots (8)$$

Now $f(0.4, 1.528) = 0.4 + 1.528$

$$f(0.4, 1.528) = 1.928 \quad \ldots (9)$$

Substituting (9) in (8) we get

$$y_2 = 1.24 + 0.1(1.44 + 1.928)$$
$$= 1.24 + 0.1(3.368)$$
$$y_2 = 1.24 + 0.3368 = 1.5768$$

$$\therefore \; y(0.4) = 1.5768$$

Example 2

Using Improved Euler's method solve $y' = x + y + xy$, $y(0) = 1$ compute y at $x = 0.1$, by taking $h = 0.1$. [A.U May '99]

Solution

Given $y' = x + y + xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

The Improved Euler's algorithm is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \ldots (1)$$

Putting $m = 0$ in (1) we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\} \quad ...(2)$$

Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y + xy$

$$\therefore f(x_0, y_0) = x_0 + y_0 + x_0 y_0 = 0 + 1 + 0(1) = 1 \quad ...(3)$$

Substituting (3) in (2) we get

$$y_1 = 1 + \frac{0.1}{2}\{1 + f[0 + 0.1, 1 + 0.1(1)]\}$$

$$y_1 = 1 + 0.05\{1 + f(0.1, 1.1)\} \quad ...(4)$$

Now $f(0.1, 1.1) = 0.1 + 1.1 + (0.1)(1.1) = 1.31 \quad ...(5)$

Substituting (5) in (4) we get

$$y_1 = 1 + 0.05(1 + 1.31)$$

$$y(0.1) = 1 + (0.05)(2.31) = 1 + 0.1155$$

$$= 1.1155$$

$$\therefore \mathbf{y(0.1) = 1.1155}$$

Example 3

$y(0.3)$

Example 11

Given that $\frac{dy}{dx} = \frac{x-y}{x+y}$, $y(2) = 1$, ***compute*** $y(1.9)$ ***by using Improved Euler's method and*** $y(1.8)$ ***by using Modified Euler's method.***

Solution

The **Improved Euler's algorithm** is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f[x_m + h, y_m + h f(x_m, y_m)]\} \quad \dots(1)$$

Putting $m = 0$ in (1), we get

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + h f(x_0, y_0)]\} \quad \dots(2)$$

Here $x_0 = 2$, $y_0 = 1$ and $h = -0.1$.

$$\text{Now } f(x_0, y_0) = \frac{x_0 - y_0}{x_0 + y_0} = \frac{2-1}{2+1} = \frac{1}{3} = 0.33333 \quad \dots(3)$$

Substituting (3) in (2), we get

$$y_1 = 1 + (-0.05)\,[0.33333 + f(2 - 0.1, 1 + (-0.1)(0.33333)]$$

$$= 1 - 0.05\,[\,0.33333 + f(1.9, 0.966667)\,] \quad \dots(4)$$

$$\text{Now } f(1.9, 0.966667) = \frac{1.9 - 0.966667}{1.9 + 0.966667} = 0.325581241 \quad \dots(5)$$

Substituting (5) in (4), we get

$$y_1 = y(1.9) = 1 - 0.05\,[\,0.333333 + 0.325581\,]$$

$$= 0.967054438$$

$$y(1.9) = 0.96705$$

The **Modified Euler's algorithm** is

$$y_{n+1} = y_n + h f\left[\,x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\,\right] \quad \ldots (6)$$

Putting $n = 0$ in (6), we get

$$y_1 = y_0 + h f\left[\,x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\,\right] \quad \ldots (7)$$

Here $x_0 = 2$; $y_0 = 1$ and $h = -0.2$

$$f(x_0, y_0) = \frac{x_0 - y_0}{x_0 + y_0}$$

$$= \frac{2-1}{2+1} = \frac{1}{3} = 0.33333 \quad \ldots (8)$$

Substituting (8) in (7), we get

$$y_1 = 1 + (-0.2)\, f\left[2 + \frac{(-0.2)}{2},\ 1 + \frac{(-0.2)}{2}(0.33333)\right]$$

$$y_1 = 1 + (-0.2)\, f(1.9, 0.966667) \quad \ldots (9)$$

Now $f(1.9, 0.966667) = \dfrac{1.9 - 0.299997}{1.9 + 0.299997} = 0.3256 \quad \ldots (10)$

Substituting (10) in (9), we get

$$y_1 = 1 + (-0.2)(0.3256)$$

$$= 0.93488$$

$$y(1.8) = 0.9349$$

THE RUNGE - KUTTA METHODS

This method was devised by Runge, about the year 1934 and extended by Kutta a few years later. Therefore we call this method as **Runge-Kutta method**. Unlike any of the methods, explained in the preceding two sections the increments of the function are calculated once for all by means of a definite set of formulae.

Here a set of formulae are given without proof for solving a differential equation of the form $\frac{dy}{dx} = f(x, y)$ under the initial condition $y(x_0) = y_0$. Let h denote the length of the interval between equidistant values of x. The various types of formulae according to their order are given below.

SECOND ORDER RUNGE - KUTTA METHOD

If the initial values are x_0, y_0 for the differential equation $\frac{dy}{dx} = f(x, y)$ then the first increment in y *viz* Δy is computed from the formulae

$k_1 = h f(x_0, y_0)$
$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
$\Delta y = k_2$

$$\text{Now } x_1 = x_0 + h, \quad y_1 = y_0 + \Delta y$$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$k_1 = h f(x_1, y_1)$,
$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$,
$\Delta y = k_2$,

and so on for the succeeding intervals.

THIRD ORDER RUNGE - KUTTA METHOD

The third order Runge-Kutta method is designed by the following formulae.

$$k_1 = h f(x_0, y_0),$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1),$$

Now the first increment in y viz Δy is computed from

$$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$\text{Now } x_1 = x_0 + h, \; y_1 = y_0 + \Delta y$$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$$k_1 = h f(x_1, y_1),$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right),$$

$$k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1),$$

$$\Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3],$$

and so on for the succeeding intervals.

FOURTH ORDER RUNGE-KUTTA METHOD

This method is most commonly used in practice. Unless and otherwise stated, Runge-Kutta method means only Fourth Order Runge-Kutta method. Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation to be solved under the condition $y(x_0) = y_0$. If h be the length of the interval between equidistant values, then the first increment in y is computed from the formulae.

$k_1 = h f(x_0, y_0)$,
$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$,
$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$k_4 = h f(x_0 + h, y_0 + k_3)$
$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Now $x_1 = x_0 + h, y_1 = y_0 + \Delta y$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$k_1 = h f(x_1, y_1)$,
$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$,
$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$,
$k_4 = h f(x_1 + h, y_1 + k_3)$,
$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and so on for the succeeding intervals.

Note : **If $\frac{dy}{dx}$ is a function of x alone then the Fourth Order Runge-Kutta method reduces to Simpson's Rule.**

For, $k_1 = h f(x_0)$
$k_2 = h f\left(x_0 + \frac{h}{2}\right)$
$k_3 = h f\left(x_0 + \frac{h}{2}\right)$
$k_4 = h f(x_0 + h)$

$$\therefore \Delta y = \frac{h}{6}\left[f(x_0) + 2f\left(x_0 + \frac{h}{2}\right) + 2f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

$$= \frac{\left(\frac{h}{2}\right)}{3}\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

which is the same result as would be obtained by applying Simpson's Rule in the interval x_0 to $x_0 + h$ if we take two equal sub-intervals of width $\frac{h}{2}$.

Now we shall illustrate this method in the following examples.

Example 1

Use Runge – Kutta method to approximate y, when x = 0.1, 0.2, 0.3, h = 0.1 given x = 0 when y = 1 and $\frac{dy}{dx} = x + y$

[A.U. Nov. '91]

Solution

Given $y' = x + y$

i.e., $f(x, y) = x + y$

And also given that $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

To find y (0.1) using third order Runge-Kutta Method :

Now $k_1 = hf(x_0, y_0)$

$= h[x_0 + y_0]$

$k_1 = (0.1)[0 + 1] = 0.1$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0 + \frac{0.1}{2} + 1 + \frac{0.1}{2}\right]$$

$= (0.1)[0.05 + 1.05]$

$k_2 = 0.11$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

$= h(x_0 + h + y_0 + 2k_2 - k_1)$

$= (0.1)\ [0 + 0.1 + 1 + 2(0.11) - 0.1]$

$= (0.1)(1.22)$

$k_3 = 0.122$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}[0.1 + 4(0.11) + 0.122] = \frac{1}{6}(0.662)$$

$\therefore \Delta y = 0.1103$

$y_1 = y_0 + \Delta y$

$= 1 + 0.1103$

$\therefore$ **$y(0.1) = 1.1103$**

To find $y(0.2)$ using third order Runge-Kutta Method :

Here $x_0 = 0.1, \quad y_0 = 1.1103, \quad h = 0.1$

Now $k_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$= (0.1)[0.1 + 1.1103]$

$k_1 = 0.12103$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0.1 + \frac{0.1}{2} + 1.1103 + \frac{0.12103}{2}\right]$$

$= (0.1)(1.3208)$

$k_2 = 0.13208$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

$= h(x_0 + h + y_0 + 2k_2 - k_1)$

$= (0.1)[0.1 + 0.1 + 1.1103 + 2(0.13208) - 0.12103]$

$= (0.1)(1.4534)$

$k_3 = 0.14534$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}\ [0.12103 + 4\ (0.13208) + 0.14534]$$

$$= \frac{1}{6}\ (0.7947)$$

$$\therefore \Delta y = 0.1324$$

$$y_2 = y_1 + \Delta y$$

$$y(0.2) = y(0.1) + \Delta y$$

$$= 1.1103 + 0.1324$$

$$\therefore\ y(0.2) = 1.2427$$

To find $y(0.3)$ using third order Runge-Kutta Method :

Here $x_0 = 0.2,\ y_0 = 1.2427,\ h = 0.1$

Now $k_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$$= (0.1)\ [0.2 + 1.2427] = (0.1)\ (1.4427)$$

$$k_1 = 0.14427$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$

$$= (0.1)\left[0.2 + \frac{0.1}{2} + 1.2427 + \frac{0.14427}{2}\right]$$

$$= (0.1)\ (1.5648)$$

$$k_2 = 0.15648$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

$$= h(x_0 + h + y_0 + 2k_2 - k_1)$$

$$= (0.1)\ [0.2 + 0.1 + 1.2427 + 2(0.15648) - 0.14427]$$

$$= (0.1)\ (1.7114)$$

$$k_3 = 0.17114$$

$$\therefore \quad \Delta y = \frac{1}{6}\ [k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}\ [0.14427 + 4\ (0.15648) + 0.17114]$$

$$= \frac{1}{6}\ (0.9413)$$

$$\therefore \Delta y = 0.1569$$

$$y_3 = y_2 + \Delta y$$

i.e., $$y(0.3) = y(0.2) + \Delta y$$

$$= 1.2427 + 0.1569$$

$$\therefore y(0.3) = 1.3996$$

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3996

Example 3

By applying the fourth order Runge-Kutta Method find y (0.2) from y′ = y − x, y (0) = 2 taking h = 0.1.

Solution

Given

$$y' = y - x$$

$$\text{i.e., } f(x, y) = y - x$$

and $y(0) = 2$ i.e., $x_0 = 0$, $y_0 = 2$ and $h = 0.1$. We know that the fourth order Runge-Kutta formula for finding the first increment in y viz Δy is given by

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$k_1 = hf(x_0, y_0)$
$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$k_4 = hf(x_0 + h, y_0 + k_3)$

$$\therefore k_1 = (0.1)(y_0 - x_0) = 0.1(2 - 0) = 0.2$$

$$k_2 = (0.1)\left[\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.1 - 0.05] = 0.205$$

$$k_3 = (0.1)\left[\left(y_0 + \frac{k_2}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.205}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.1025 - 0.05]$$

$$= 0.20525$$

$$k_4 = (0.1)[(y_0 + k_3) - (x_0 + h)]$$

$$= (0.1)[2 + 0.20525 - 0 - 0.1]$$

$$= 0.210525$$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2 + 2(0.205) + 2(0.20525) + 0.210525]$$

$$= \frac{1}{6}[0.2 + 0.41 + 0.4105 + 0.210525]$$

$$= 0.20517$$

$$\therefore y(0.1) = y_1 = y_0 + \Delta y$$

$$= 2 + 0.20517$$

$$\therefore \mathbf{y(0.1) = 2.20517}$$

Next we have to find $y(0.2) = y_2 = y_1 + \Delta y$

where $\boxed{\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]}$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

Now $\boxed{k_1 = hf(x_1, y_1)} = h[y_1 - x_1]$

$= (0.1)[2.20517 - 0.1] = 0.210517$

$$\boxed{k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)}$$

$$= h\left[\left(y_1 + \frac{k_1}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2.20517 + \frac{0.2105}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.31042 - 0.15] = 0.21604$$

$$\boxed{k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)}$$

$$= h\left[\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2.20517 + \frac{0.21604}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.31319 - 0.15] = 0.21632$$

$$\boxed{k_4 = hf(x_1 + h, y_1 + k_3)}$$

$$= h[(y_1 + k_3) - (x_1 + h)]$$

$$= (0.1)[(2.20517 + 0.21632) - (0.1 + 0.1)]$$

$$= (0.1)[2.142149 - 0.2] = 0.22214$$

$$\boxed{\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]}$$

$$= \frac{1}{6}[0.2105 + 2(0.21604) + 2(0.21632) + 0.22214]$$

$$= \frac{1}{6}[0.2105 + 0.43208 + 0.43264 + 0.22214]$$

$$= 0.21622$$

$\therefore y_2 = y_1 + \Delta y$

$= 2.20517 + 0.21622$

$y(0.2) = 2.42139$

Hence we have the following table.

x	0	0.1	0.2
y	2	2.20517	2.42139

Example 4

Find the values of y(0.2) and y(0.4) using Runge-Kutta Method of fourth order with h = 0.2, given that $\frac{dy}{dx} = \sqrt{x^2 + y}$; $y(0) = 0.8$.

Solution

Given
$$y' = \sqrt{x^2 + y}$$
$$\text{i.e., } f(x, y) = \sqrt{x^2 + y}$$

And also given that $x_0 = 0$, $y_0 = 0.8$ and $h = 0.2$.

To find $y(0.2)$:

We know that the fourth order Runge-Kutta formula to find the first increment in y viz. Δy is given by

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = hf(x_0, y_0) = h\left[\sqrt{x_0^2 + y_0}\right]$

$$= (0.2)\left[\sqrt{0 + 0.8}\right]$$

$$k_1 = 0.17889$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= h\left[\sqrt{\left(x_0 + \frac{h}{2}\right)^2 + \left(y_0 + \frac{k_1}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0 + \frac{0.2}{2}\right)^2 + \left(0.8 + \frac{0.17889}{2}\right)}\right]$$

$$= (0.2)\sqrt{(0.1)^2 + 0.8 + 0.08944}$$

$$k_2 = 0.18968$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$

$$= h\left[\sqrt{\left(x_0 + \frac{h}{2}\right)^2 + \left(y_0 + \frac{k_2}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0 + \frac{0.2}{2}\right)^2 + \left(0.8 + \frac{0.18968}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{(0.1)^2 + 0.8 + 0.09484}\right]$$

$$k_3 = 0.19025$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h\sqrt{(x_0 + h)^2 + (y_0 + k_3)}$$

$$= h\sqrt{(0 + 0.2)^2 + (0.8 + 0.19025)}$$

$$= (0.2)\sqrt{0.04 + 0.99025}$$

$$k_4 = 0.20300$$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.17889 + 2(0.18968) + 2(0.19025) + 0.20300]$$

$$= \frac{1}{6}[1.14175]$$

$$\Delta y = 0.19029$$

$$\therefore y(0.2) = y_0 + \Delta y$$

$$= 0.8 + 0.19029$$

$$\mathbf{y(0.2) = 0.99029}$$

To find $y(0.4)$

Here $x_1 = 0.2$, $y_1 = 0.99029$ and $h = 0.2$

Now $k_1 = hf(x_1, y_1) = h\left[\sqrt{x_1^2 + y_1}\right]$

$$= (0.2)\left[\sqrt{(0.2)^2 + 0.99029}\right]$$

$$k_1 = 0.20301$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right]$$

$$= h\left[\sqrt{\left(x_1 + \frac{h}{2}\right)^2 + \left(y_1 + \frac{k_1}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0.2 + \frac{0.2}{2}\right)^2 + \left(0.99029 + \frac{0.20301}{2}\right)}\right]$$

$$= (0.2)\sqrt{(0.3)^2 + 0.99029 + 0.10150}$$

$$k_2 = 0.21742$$

$$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$$

$$= h\left[\sqrt{\left(x_1 + \frac{h}{2}\right)^2 + \left(y_1 + \frac{k_2}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{\left(0.2 + \frac{0.2}{2}\right)^2 + \left(0.99029 + \frac{0.21742}{2}\right)}\right]$$

$$= (0.2)\left[\sqrt{(0.3)^2 + 0.99029 + 0.10871}\right]$$

$$k_3 = 0.21808$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h\sqrt{(x_1 + h)^2 + (y_1 + k_3)}$$

$$= h\sqrt{(0.2 + 0.2)^2 + (0.99029 + 0.21808)}$$

$$= (0.2)\sqrt{1.36837} = 0.23396$$

$$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.20301 + 2(0.21742) + 2(0.21808) + 0.23396$$

$$= \frac{1}{6}[1.30797] = 0.217996$$

$$\therefore y_2 = y_1 + \Delta y$$

$$= 0.99029 + 0.217996$$

$$\therefore y(0.4) = 1.20828$$

x	0	0.2	0.4
y	0.8	0.99029	1.20828