TRAPEZOIDAL RULE

Putting $n = 1$ in (A), we get

$$\int_{x_0}^{x_0+h} y(x)\, dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

(neglecting higher order differences)

$$= \frac{h}{2}\,[2y_0 + \Delta y_0] = \frac{h}{2}\,[y_0 + (y_0 + \Delta y_0)]$$

$$= \frac{h}{2}\,[y_0 + y_1] \qquad \ldots (1)$$

In the interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} y(x)\, dx = h\,[y_1 + \frac{1}{2}\,\Delta y_1]$$

$$= \frac{h}{2}\,[2y_1 + \Delta y_1] = \frac{h}{2}\,[y_1 + (y_1 + \Delta y_1)]$$

$$= \frac{h}{2}\,[y_1 + y_2] \qquad \ldots (2)$$

....

.... etc

$$\int_{x_0+(n-1)h}^{x_0+nh} y(x)\, dx = \frac{h}{2}\,[y_{n-1} + y_n] \qquad \ldots (3)$$

Adding (1), (2) and (3), we get

$$\int_{x_0}^{x_0+nh} y(x)\, dx = \frac{h}{2}\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots + y_{n-1})] \qquad \ldots (B)$$

This is called the **Trapezoidal Rule.**

Note: The trapezoidal rule is the simplest of the formulae for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval h.

TRUNCATION ERROR IN THE TRAPEZOIDAL RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \quad \ldots (1)$$

where y_1 is the value of y at $x = x_1$ and y_1', y_1''... etc are the values of y', y'', etc at $x = x_1$.

$$\therefore \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} \left[y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x - x_1)^2}{2!} y_1' + \frac{(x - x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_2}$$

$$= y_1 (x_2 - x_1) + \frac{(x_2 - x_1)^2}{2!} y_1' + \frac{(x_2 - x_1)^3}{3} y_1'' + \ldots$$

$$= h y_1 + \frac{h^2}{2!} y_1' + \frac{h^3}{3!} y_1'' + \quad \ldots (2)$$

where $h = x_2 - x_1$

Now, A_1 = area of the trapezium in the interval (x_1, x_2)

$$= \frac{1}{2} h (y_1 + y_2) \quad \ldots (3)$$

Putting $x = x_2$ and $y = y_2$ in (1), we get

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \ldots \quad \ldots (4)$$

where $h = x_2 - x_1$

Substituting (4) in (3), we get

$$A_1 = \frac{h}{2}\left[2y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \dots\right]$$

$$= hy_1 + \frac{h^2}{2!}y_1' + \frac{h^3}{2 \times 2!}y_1'' + \dots \qquad \dots(5)$$

(2) − (5) ⇒

$$\int_{x_1}^{x_2} y\,dx - A_1 = \left(\frac{1}{3!} - \frac{1}{2 \times 2!}\right) h^3 y_1'' + \dots$$

$$= \frac{-h^3}{12} y_1'' + \dots$$

i.e., Principal part of the error in (x_1, x_2)

$$= \frac{-h^3}{12} y_1''$$

Similarly principal part of the error in the interval (x_2, x_3)

$$= \frac{-h^3}{12} y_2'' \text{ and so on.}$$

Hence the total error $E = \frac{-h^3}{12}\left[y_1'' + y_2'' + \dots + y_n''\right]$

$$\therefore E < \frac{-nh^3}{12} y''(\xi)$$

Where $y''(\xi)$ is the largest of the n quantities $y_1'', y_2'', \dots, y_n''$.

i.e., $E < \frac{-nh^3}{12} y''(\xi) = -\frac{(b-a)\,h^2}{12} y''(\xi) \quad \left[\because n = \frac{b-a}{h}\right]$

∴ **Error in the trapezoidal rule is of the order h^2.**

Example 1

Compute the value of the definite integral $\int_4^{5.2} \log_e x\, dx$ ***or*** $\int_4^{5.2} \ln x\,dx$ ***using trapezoidal rule.***

Solution

Divide the interval of integration into six equal parts each of width 0.2 *i.e.*, $h = 0.2$. The values of the function $y = \ln x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\ln x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By **Trapezoidal rule,** we have

$$\int_4^{5.2} \ln x\,dx = \frac{h}{2}\,[(y_0 + y_6) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.2}{2}[(1.386294 + 1.648658) + 2\,(1.435084 + 1.481604 + 1.526056 + 1.568616 + 1.609437)]$$

$$= (0.1)\,[\,3.034952 + 15.241562\,]$$

$$\int_4^{5.2} \ln x\,dx = \mathbf{1.8276544}$$

Example 2

Evaluate $\int_0^1 e^{-x^2}\,dx$ ***by dividing the range of integration into 4 equal parts using trapezoidal rule.*** *[Nov. '91, Nov. '89]*

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By **Trapezoidal rule** we have

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{2}\,[\,(y_0 + y_4) + 2\,(y_1 + y_2 + y_3)\,]$$

$$= \frac{0.25}{2}[1.3678 + 2\,(2.2876)] = (0.125)\,(5.943)$$

$$\int_0^1 e^{-x^2}\,dx = \mathbf{0.7428}$$

❑ SIMPSON'S $\frac{1}{3}$ RULE

Putting $n = 2$ in the above relation (A) (Refer Pg. No. 3.32) and neglecting all differences above the second we get,

$$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6(y_1 - y_0) + y_2 - 2y_1 + y_0}{6}\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\therefore \int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2] \quad \ldots (1)$$

Similarly for the next two intervals $x_0 + 2h$ to $x_0 + 4h$ we get,

$$\int_{x_0+2h}^{x_0+4h} y(x)\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4] \quad \ldots (2)$$

In general,

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\,dx = \frac{h}{3}\,[\,y_{n-2} + 4y_{n-1} + y_n\,] \quad \ldots (3)$$

Adding all the above integrals (1), (2), (3) we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[y_0 + 4\,(y_1 + y_3 + \ldots) + 2\,(y_2 + y_4 + \ldots) + y_n]$$

$$= \frac{h}{3}\,[y_0 + y_n + 4\,(\text{sum of odd ordinates}) + 2\,(\text{sum of even ordinates})]$$

This is called Simpson's one third rule or Simpson's $\frac{1}{3}$ rule.

Note 1 : **When using this formula the student must bear in mind that the interval of integration must be divided into an even number of subintervals of width h.**

Note 2 : **Simpson's $\frac{1}{3}$ rule is also called a closed formula, since the end point y_0 and y_1 are also included in the formula.**

❑ SIMPSON'S THREE – EIGHTH RULE :

Putting $n = 3$ in (A) (Refer Pg. No. 3.32) and neglecting the higher order differences above the third we get

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2\,(y_3 + y_6 + \ldots + y_{n-3})].$$

This is known as **Simpson's three - eighth rule.**

Note : **This rule can be applied only if the number of subintervals is a multiple of 3.**

❑ TRUNCATION ERROR IN SIMPSON'S RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x-x_1)}{1!} y_1' + \frac{(x-x_1)^2}{2!} y_1'' + \ldots \quad \ldots (1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'', \ldots$ *etc.* are the values of $y', y'', \ldots$ *etc.* at $x = x_1$.

Hence

$$\int_{x_1}^{x_3} y\,dx = \int_{x_1}^{x_3} \left[y_1 + \frac{(x-x_1)}{1!} y_1' + \frac{(x-x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x-x_1)^2}{2!} y_1' + \frac{(x-x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_3}$$

$$= y_1 (x_3 - x_1) + \frac{(x_3-x_1)^2}{2!} y_1' + \frac{(x_3-x_1)^3}{3!} y_1'' + \ldots$$

$$= 2hy_1 + \frac{(2h)^2}{2!} y_1' + \frac{(2h)^3}{3!} y_1'' + \frac{(2h)^4}{4!} y_1''' + \frac{(2h)^5}{5!} y_1^{iv} + \ldots$$

$$[\because x_2 - x_1 = h; \therefore x_3 - x_1 = 2h]$$

$$= 2hy_1 + 2h^2 y_1' + \frac{4h^3}{3} y_1'' + \frac{2h^4}{3} y_1''' + \frac{4h^5}{15} y_1^{iv} + \ldots \quad \ldots (2)$$

Now, Area A_1 = area over the first double strip by Simpson's $\frac{1}{3}$ rule

$$= \frac{1}{3} h (y_1 + 4y_2 + y_3) \quad \ldots (3)$$

Putting $x = x_2$ and therefore $y = y_2$ in (1), we get,

$$y_2 = y_1 + \frac{(x_2-x_1)}{1!} y_1' + \frac{(x_2-x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \quad \ldots (4)$$

where $h = x_2 - x_1$

Putting $x = x_3$ and therefore $y = y_3$ in (1), we get,

$$y_3 = y_1 + \frac{(x_3-x_1)}{1!} y_1' + \frac{(x_3-x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{2h}{1!} y'_1 + \frac{4h^2}{2!} y_1'' + \frac{8h^3}{3!} y_1''' + \frac{16h^4}{4!} y_1^{iv} + \ldots \quad \ldots (5)$$

Substituting (4) and (5) in (3), we get

$$A = \frac{h}{3}\left[y_1 + 4\left\{ y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \right\} \right.$$

$$\left. + \left\{ y_1 + \frac{2h}{1!} + \frac{4h^2}{2!} y_1'' + \frac{8h^3}{3!} y_1''' + \frac{16h^4}{4!} y_1^{iv} + \ldots \right\} \right] \quad \ldots (6)$$

Subtracting (2) and (6), we get

$$\int_{x_1}^{x_3} y\,dx - A_1 = \left[\frac{4h^5}{15} y_1^{iv} - \frac{5h^5}{18} y_1^{iv} \right] + \ldots$$

$\therefore$ The error in the interval (x_1, x_3),

$$= \left(\frac{4}{15} - \frac{5}{18} \right) h^5 y_1^{iv} = \left(\frac{24 - 25}{90} \right) h^5 y^{iv}$$

$$= \frac{-h^5}{90} y_1^{iv}$$

$\therefore$ The principal part of the error in (x_1, x_3)

$$= \frac{-h^5}{90} y_1^{iv}$$

Similarly, principal part of the error in the interval (x_3, x_5),

$$= \frac{-h^5}{90} y_3^{iv} \text{ and so on.}$$

Hence the total error E

$$= \frac{-h^5}{90} y_1^{iv} - \frac{h^5}{90} y_3^{iv} - \ldots - \frac{h^5}{90} y_{2n-1}^{iv}$$

$$= \frac{-h^5}{90} \left[y_1^{iv} + y_3^{iv} + \ldots + y_{2n-1}^{iv} \right]$$

$$= \frac{-nh^5}{90} y^{iv}(\xi)$$

where $y^{iv}(\xi)$ is the largest of the n quantities $y_1^{iv}, y_3^{iv}, \ldots y_{2n-1}^{iv}$.

i.e., $$E < \frac{-(b-a)}{2h} \cdot \frac{h^5}{90} \quad \left[\because \frac{b-a}{2n} = h, \text{ i.e., } h = \frac{b-a}{2h} \right]$$

$$< -\frac{h^4}{180}(b-a)$$

∴ **Error in the Simpson's $\frac{1}{3}$ rule is of the order h^4.**

Example 1

Compute the value of the definite integral $\int_4^{5.2} \log_e x \, dx$ or $\int_4^{5.2} \ln x dx$ using Simpson's rule.

Solution

Divide the interval of integration into six equal parts each of width 0.2 *i.e.,* $h = 0.2$. The values of the function $y = ln\ x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$ln\ x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_4^{5.2} \ln x\, dx = \frac{h}{3}\left[(y_0 + y_6) + 2\,(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{0.2}{3}\ [3.034952 + 2\,(3.050221) + 4\,(4.570577)]$$

$$\int_4^{5.2} \ln x\, dx = \mathbf{1.827847}$$

Example 2

Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal parts using Simpson's rule. *[Nov. '91, Nov. '89]*

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By Simpson's rule we have

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{3}\left[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3) \right]$$

$$= \frac{0.25}{3}\left[1.3678 + 1.5576 + 6.0368\right]$$

$$\int_0^1 e^{-x^2}\,dx = 0.7468$$

Example 10

Evaluate $\int_1^4 f(x)\,dx$ *from the following table by Simpson's* $\frac{3}{8}$ *rule.*

x	1	2	3	4
$y = f(x)$	1 y_0	8 y_1	27 y_2	64 y_3

Solution

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_1^4 f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$= \frac{3(1)}{8}\left[1 + 3(8) + 3(27) + 64\right]$$

$$= \frac{3}{8}\left[1 + 24 + 81 + 64\right] = \frac{3}{8}[170]$$

$$\int_1^4 f(x)\,dx = 63.75$$

Example 11

By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x \, dx$ by using Simpson's $\frac{1}{3}$ rule. Is it possible to evaluate the same by Simpson's $\frac{3}{8}^{th}$ rule. Justify your answer.

Solution

Here range $= \pi - 0 = \pi$

$$h = \frac{\pi}{10}$$

The values of the function $y = \sin x$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
y	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3}[(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$= \frac{\frac{\pi}{10}}{3}[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1.0 + 0.8090 + 0.3090)]$$

$$\int_0^{\pi} \sin x \, dx = 2.00091$$

Note : Here we cannot use Simpson's $\frac{3}{8}^{th}$ rule since the subintervals is not a multiple of 3.

GAUSS QUADRATURE FORMULA

Carl Frederich Gauss approached the problem of numerical integration in a different way. Instead of finding the area under the given curve, he tried to evaluate the function at some points along with the abscissa. Here the values of abscissa are not equal. Then apply certain weight to the evaluated function.

Thus for Gauss two point formula,

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

$$= \omega_1 f(t_1) + \omega_2 f(t_2) \quad \ldots (1)$$

The function $f(t)$ is evaluated at t_1 and t_2. ω_1 and ω_2 are the weights given to the two functions.

The basic methodology is explained as given below for **Gauss two point formula.**

GAUSS - TWO POINT FORMULA

First change the interval (a, b) to $(-1, 1)$ by using the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

Thus the independent variable 'x' is changed to 't'.

Then we use an interpolation formula which will give the true value of the integral at certain points. Here the interpolation points are t_1 and t_2.

In equation (1), we want to find the four unknown quantities ω_1, ω_2 and t_1 , t_2. So we need four algebraic equations to solve it. Let the equation (1) be exact for

$$f(t) = 1$$

$$f(t) = t$$

$$f(t) = t^2 \text{ and } f(t) = t^3$$

Now $\quad f(1) = 1$

$$\Rightarrow \int_{-1}^{1} 1\,dt = 2 = \omega_1 + \omega_2 \quad [\because f(t_1) = f(t_2) = 1] \ldots (2)$$

$$f(t) = t$$

$$\Rightarrow \int_{-1}^{1} t\,dt = \left(\frac{t^2}{2}\right)_{-1}^{1} = 0 = \omega_1 t_1 + \omega_2 t_2 \quad \ldots(3)$$

$$f(t) = t^2$$

$$\Rightarrow \int_{-1}^{1} t^2\,dt = \left(\frac{t^3}{3}\right)_{-1}^{1} = \frac{2}{3}$$

$$= \omega_1 t_1^2 + \omega_2 t_2^2 \quad \ldots(4)$$

$$f(t^2) = \int_{-1}^{1} t^3\,dt$$

$$\Rightarrow \left(\frac{t^4}{4}\right)_{-1}^{1} = 0 = \omega_1 t_1^3 + \omega_2 t_2^3 \quad \ldots(5)$$

This set of equations (2), (3), (4) and (5) can be solved as follows.

From (3), we get

$$\omega_1 t_1 = -\omega_2 t_2 \quad \ldots(6)$$

From (5), we get

$$\omega_1 t_1^3 = -\omega_2 t_2^3 \quad \ldots(7)$$

From (6) and (7), we get

$$t_1 = -t_2$$

$$\omega_1 = \omega_2 = 1$$

From (4), we get $t_1^2 + t_2^2 = \frac{2}{3}$

$$\Rightarrow \quad t_1 = \frac{1}{\sqrt{3}}; \quad t_2 = \frac{-1}{\sqrt{3}}$$

From equation (1), we get

$$I = \int_{-1}^{1} f(t)\,dt = \omega_1 f(t_1) + \omega_2 f(t_2)$$

$$\boxed{I = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)} \quad \ldots(A)$$

$[\because \omega_1 = \omega_2 = 1]$

Evaluate $\int_1^2 \frac{dx}{x}$ *using Gauss 2 point formula.*

Solution

Transform the variable x to t by the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$= \left(\frac{1+2}{2}\right) + \left(\frac{2-1}{2}\right)t$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2}$$

$$\text{i.e.,} \quad dx = \frac{dt}{2}$$

$$\therefore \text{I} = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{2}{3+t} \cdot \frac{dt}{2} = \int_{-1}^{1} \frac{dt}{3+t}$$

Here $$f(t) = \frac{1}{3+t} \quad \text{... (A)}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3+\frac{1}{\sqrt{3}}} = 0.2795$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{3-\frac{1}{\sqrt{3}}} = 0.41288$$

$$\text{I} = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right),$$

where $f(t) = \frac{1}{3+t}$ [By (A)]

$$\textbf{I} = \textbf{0.6923}$$

Example 2

Evaluate $\int_{1}^{2} \frac{dx}{1+x^3}$ *using Gaussian 2 point formula.*

Solution

Transform the variable x to t by

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2} \qquad \text{... (A)}$$

$$dx = \frac{dt}{2}$$

$$\therefore \text{I} = \int_{1}^{2} \frac{dx}{1+x^3} = \int_{-1}^{1} \frac{1}{1+\left(\frac{3+t}{2}\right)^3} \frac{dt}{2}$$

$$= 4 \int_{-1}^{1} \frac{dt}{8+(3+t)^3} = 4\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right],$$

Here $f(t) = \frac{1}{8+(3+t)^3}$ [By (A)]

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{8+\left((3+\frac{1}{\sqrt{3}})\right)^3} = 0.0185$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{8+\left((3-\frac{1}{\sqrt{3}})\right)^3} = 0.045$$

$$= 4\,[\,0.0185 + 0.045\,]$$

$$\text{I} = 0.254$$

☐ GAUSSIAN QUADRATURE (3 POINT) FORMULA

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

where the interval (a, b) is changed into $(-1, 1)$ by the transformation,

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

Then $$\int_{-1}^{1} f(t)\,dt = A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)$$

where
$$A_1 = A_3 = 0.5555$$
$$A_2 = 0.8888$$
$$t_1 = -0.7745$$
$$t_2 = 0$$
$$t_3 = 0.7745$$

Example 1

Evaluate $\int_1^2 \frac{dx}{x}$ *using Gauss 3 - point formula.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$
$$= \frac{3}{2} + \frac{t}{2}$$

i.e., $$x = \frac{3+t}{2}$$

$$\therefore I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} f(t)\,dt$$

$$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \quad \ldots (1)$$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots (2)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{3-0.7745} = 0.4493 \\ f(t_2) &= f(0) = \frac{1}{3} = 0.3333 \\ f(t_3) &= f(0.7745) = \frac{1}{3+0.7745} = 0.2649 \end{aligned}\right\} \quad \dots (3)$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.4493) + 0.8888\,(0.3333) + (0.2649)\,(0.5555)$$

$$\boxed{I = 0.6929}$$

Example 2

Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ *using the three point Gaussian Quadrature.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

where $a = 0.2$, $b = 1.5$

$$= \frac{1.7}{2} + \frac{1.3t}{2}$$

i.e., $\quad x = \dfrac{1.7 + 1.3\,t}{2} \Rightarrow dx = \dfrac{1.3\,dt}{2} = 0.65\,dt$

$$\therefore \quad I = \int_{0.2}^{1.5} e^{-x^2}\,dx$$

$$= \int_{-1}^{1} -e^{\left(\frac{1.7+1.3\,t}{2}\right)^2} (0.65)\,dt$$

$$= 0.65 \int_{-1}^{1} -e^{\left(\frac{1.7+1.3\,t}{2}\right)^2} dt \quad \dots (1)$$

$$I = 0.65[A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)] \quad \dots (2)$$

where $f(t) = e^{-\left(\frac{1.7 + 1.3\,t}{2}\right)^2}$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots (3)$$

$$f(t_1) = f(-0.7745) = e^{-\left(\frac{1.7 + 1.3(-0.7745)}{2}\right)^2}$$
$$= 0.8868$$

$$f(t_2) = f(0) = e^{-\left(\frac{1.7 + 1.3\,(0)}{2}\right)^2}$$
$$= 0.48555 \quad \ldots (4)$$

$$f(t_3) = f(0.7745) = e^{-\left(\frac{1.7 + 1.3\,(0.7745)}{2}\right)^2}$$
$$= 0.16013$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.8868) + 0.8888\,(0.4855) + (0.5555)(0.16013)$$

$$= 0.4926 + 0.4315 + 0.08895$$

$I = 1.01307$