Solutions of linear algebraic equations

A system of m linear equations (or a set of m simultaneous linear equations) in 'n' unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots (1)$$

where the coefficients of $x_1, x_2, \ldots, x_n$ and $b_1, b_2, \ldots, b_m$ are constants.

The left hand side members of (1) may be specified by the square array of the coefficients, known as the coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

whereas the complete set may be specified by the rectangular array

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{bmatrix}$$

known as the **augmented matrix.**

The solution of (1) is a set of numbers $x_1, x_2, \ldots, x_n$ which satisfy all the m equations in (1).

If the number of simultaneous equations is small (viz. 3 or 4) Crammer's rule for solving simultaneous equations will give a satisfactory result for the unknowns. However for large number of equations this rule will not give a satisfactory result. So we adopt some other methods to solve a system of linear simultaneous equations. There are two methods to solve such a system by numerical methods.

1. **Direct methods**
2. **Iterative or indirect methods.**

Gaussian elimination method, Gauss-Jordan method belongs to Direct methods, Gauss Jacobi and Gauss-Seidel iterative methods and belongs to iterative methods.

❑ Back Substitution

Consider the following linear system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ -2x_2 - x_3 &= -7 \\ -5x_3 &= -15 \end{aligned}$$

From the last equation, $x_3 = \dfrac{b_3}{a_{33}} = \dfrac{15}{5} = 3$

With this, from the second equation,

$$x_2 = \frac{-7 + x_3}{-2} = \frac{(-7+3)}{-2} = 2$$

Hence, from the first equation,

$$x_1 = \frac{5 - 3x_2 + x_3}{2} = \frac{(5 - 3.2 + 3)}{2} = 1$$

This method of obtaining the values of x_1, x_2 and x_3 is called Back Substitution method.

❑ Gauss elimination method

Basically the most effective direct solution techniques currently being used are applications of Gauss elimination, method which Gauss proposed over a century ago. In this method the given system is transformed into an equivalent system with upper triangular coefficient matrix *i.e.*, a matrix in which all elements below the diagonal elements are zero which can be solved by back substitution. This method is very clear from the following example.

Solve the following system by Gaussian elimination method

$$x_1 - x_2 + x_3 = 1$$
$$-3x_1 + 2x_2 - 3x_3 = -6$$
$$2x_1 - 5x_2 + 4x_3 = 5$$

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Solution

Step 1 : Write the given system in augmented matrix form.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)$$

Step 2 : From the first column with non-zero components (called the pivot column), select the component with the largest **absolute value.** This component is called the **pivot.**

$$\begin{array}{r} \\ \textit{Pivot} \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)$$

Step 3 : Rearrange the rows to move the pivot element to the top of First column. Here we interchange the first and second row.

$$\begin{array}{r} \textit{Pivot} \rightarrow \\ \\ \\ \end{array} \left(\begin{array}{ccc|c} -3 & 2 & -3 & -6 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right) \qquad \boxed{\mathbf{R_1 \sim R_2}}$$

Step 4 : Make the pivot as 1, by dividing the first row by the pivot.

$$\left(\begin{array}{ccc|c} 1 & \frac{-2}{3} & 1 & 2 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right) \qquad \frac{\mathbf{R_1}}{\mathbf{3}}$$

Step 5 : Add multiples of the first row to the other rows to make all the other components in the pivot column equal to zero.

$$\left(\begin{array}{ccc|c} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-1}{3} & 0 & -1 \\ 0 & \frac{-11}{3} & 2 & 1 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

Step 6 : Delete the first row and first column and perform steps 2 to 5 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-1}{3} & 0 & -1 \\ 0 & \frac{-11}{3} & 2 & 1 \end{pmatrix} \qquad \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & \frac{-11}{3} & 2 & 1 \\ 0 & \frac{-1}{3} & 0 & -1 \end{pmatrix}$$

R_3 and R_2 are interchanged to move the pivot to the top of new submatrix

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & \frac{-1}{3} & 0 & -1 \end{pmatrix}$$

R_3 is divided by $\frac{-11}{3}$ to make the pivot as 1

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & \frac{-2}{11} & \frac{-12}{11} \end{pmatrix} \qquad R_3 \to R_3 + \frac{1}{3} R_2$$

Step 7 : Delete first two rows and first two columns. Perform step 6 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & \frac{-2}{11} & \frac{-12}{11} \end{pmatrix} \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{-2}{3} & 1 & 2 \\ 0 & 1 & \frac{-6}{11} & \frac{-3}{11} \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

R_3 is divided by the new pivot.

Step 8 : Use back substitution to find the solution to the system.

Here
$$x_3 = 6$$

$$x_2 - \frac{6}{11}(6) = \frac{-3}{11}$$

i.e.,
$$x_2 = \frac{-3}{11} + \frac{6}{11}(6) = 3$$

$$x_1 - \frac{2}{3}x_2 + x_3 = 2$$

i.e.,
$$x_1 = 2 + \frac{2}{3}x_2 - x_3 = 2 + \frac{2}{3}(3) - 6 = -2$$

Hence the solution is
$$x_1 = -2, \; x_2 = 3, \; x_3 = 6.$$

Checking :

$$x_1 - x_2 + x_3 = 1$$
$$-2 - 3 + 6 = 1$$

in the top of the

Note 2 : If we are not interested in the elimination of x, y, z in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of the position of the variables.

Example 2

Solve the system of equations

$$\begin{aligned} 28x + 4y - z &= 32 \\ x + 3y + 10z &= 24 \\ 2x + 17y + 4z &= 35 \end{aligned}$$

using Gauss elimination method. *[Nov. 90 Civil, Apr. 92 ECE*

Solution

Step 1 : Write the given system in augmented matrix form

$$\begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix}$$

Step 2 : From the first column with non-zero components (called the pivot column) select the component with the largest absolute value. This component is called the pivot.

$$\text{Pivot} \rightarrow \begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix}$$

Step 3 : Make the pivot as 1 by dividing the first row by pivot.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix} \qquad R_1 \rightarrow R_1 \div 28$$

Step 4 : Add multiples of the first row to the other rows to make all the other components in the pivot column equal to zero.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \end{pmatrix} \qquad \begin{aligned} R_2 &\to R_2 - R_1 \\ R_3 &\to R_3 - 2R_1 \end{aligned}$$

Step 5 : Delete the first row and first column and perform steps 2 to 4 on the resulting matrix.

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$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \end{pmatrix} \qquad \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & \frac{117}{7} & \frac{57}{14} & \frac{229}{7} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \end{pmatrix}$$

R_3 and R_2 are interchanged to move the pivot to the top of new submatrix

14.25 ×

0.0982

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & \frac{20}{7} & \frac{281}{28} & \frac{160}{7} \end{pmatrix} \qquad R_2 \to R_2 \times \frac{7}{117}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & \frac{1457}{156} & \frac{2020}{117} \end{pmatrix} \qquad R_3 \to R_3 - \frac{20}{7} R_2$$

Step 6 : Delete first two rows and two columns. Perform step 5 on the resulting matrix.

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & \frac{1457}{156} & \frac{2020}{117} \end{pmatrix} \leftarrow \text{New Pivot}$$

$$\begin{pmatrix} 1 & \frac{1}{7} & \frac{-1}{28} & \frac{8}{7} \\ 0 & 1 & \frac{19}{78} & \frac{229}{117} \\ 0 & 0 & 1 & 1.8485 \end{pmatrix} \qquad R_3 \to R_3 \times \frac{156}{1457}$$

Step 7 : Use back - substitution to find the solution to the system.

$$z = 1.8485$$

$$y + \frac{19}{78} z = \frac{229}{117}$$

$$i.e., \ y = \frac{229}{117} - \frac{19}{78} \times 1.8485 = 1.50697$$

$$x + \frac{1}{7} y - \frac{1}{28} z = \frac{8}{7}$$

$$x = \frac{8}{7} - \frac{1}{7}(1.50697) + \frac{1}{28}(1.8485)$$

$$x = 0.9936$$

$$\therefore \ x = 0.9936 \ ; \ y = 1.5070, \ z = 1.8485$$

Checking :

$$28x + 4y - z = 32$$

$$28\,(0.9936) + 4\,(1.5070) - 1.8425 = 32.0063$$

Gauss - Jordan Method

This method is a modified form of Gaussian elimination method. In this method, the coefficient matrix is reduced to a diagonal matrix (or even a unit matrix) rather than a triangular matrix as in the Gaussian method. Here the elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal. Here we get the solution without using the back substitution method since after completion of the Gauss-Jordan method the equations become

$$\begin{bmatrix} 1\,0\,0 \ldots 0 \\ 0\,1\,0 \ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ 0\,0\,0 \ldots 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ \ldots \\ a_n \end{bmatrix}$$

Note 1 : **This method involves more computation than in the Gaussian method.**

Note 2 : **In this method we can find the values of $x_1, x_2, \ldots$ immediately without using back substitution. The method explained by the following examples.**

Example

Using Gauss – Jordan method solve the following equations.

$$5x + 4y = 15$$

$$3x + 7y = 12$$

[A.U. May '99]

Solution

Step 1 : Write the given system of equations in augmented matrix form.

$$\begin{pmatrix} 5 & 4 & 15 \\ 3 & 7 & 12 \end{pmatrix}$$

Step 2 : Make the element in the first row and first column as 1.

$$\begin{pmatrix} 1 & \frac{4}{5} & 3 \\ 3 & 7 & 12 \end{pmatrix} \quad R_1 \to R_1 \div 5$$

2

1

Step 3 : Add multiples of the first row to the other rows to make all the other components in the first column equal to zero.

$$\begin{pmatrix} 1 & \frac{4}{5} & 3 \\ 0 & \frac{23}{5} & 3 \end{pmatrix} \quad R_2 \to R_2 - 3R_1$$

Step 4 : Make the element in the second row and second column as 1.

$$\begin{pmatrix} 1 & \frac{4}{5} & 3 \\ 0 & 1 & \frac{15}{23} \end{pmatrix} \quad R_2 \to R_2 \div \frac{23}{5}$$

Step 5 : Add multiples of the second row to the other ro to make all the other components in the seco column equal to zero.

$$\begin{pmatrix} 1 & 0 & \frac{57}{23} \\ 0 & 1 & \frac{15}{23} \end{pmatrix} \quad \boxed{R_1 \to R_1 - \frac{4}{5} R_2}$$

The matrix finally reduces to the form given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{57}{23} \\ \frac{15}{23} \end{pmatrix}$$

$\therefore$

$$x = \frac{57}{23} = 2.4783$$

and

$$y = \frac{15}{23} = 0.6522$$

Checking :

$$5\,(2.4783) + 4\,(0.6522) = 15$$
$$12.3915 + 2.6088 = 15$$
$$15.000 = 15$$

Example

Solve $x + 3y + 3z = 16,\ x + 4y + 3z = 18,\ x + 3y + 4z =$ *Gauss – Jordan method.* *[A.U. Ma*

Solution

Given

$$x + 3y + 3z = 16$$
$$x + 4y + 3z = 18$$
$$x + 3y + 4z = 19$$

Step 1 : Write the given system of equations in aug matrix form.

$$\begin{pmatrix} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{pmatrix}$$

Step 2 : Add multiples of the first row to the other rows to make all the other components in the first column equal to zero.

$$\begin{pmatrix} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

Step 3 :

$$\begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad R_1 \to R_1 - 3R_2$$

Step 4 :

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad R_1 \to R_1 - 3R_3$$

he matrix finally reduces to the form given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

herefore $x = 1, y = 2, z = 3$

Checking :

$$\begin{aligned} x + 3y + 3z &= 16 \\ 1 + 3(2) + 3(3) &= 16 \\ 1 + 6 + 9 &= 16 \\ 16 &= 16 \end{aligned}$$

Iterative Methods

Examp

The previous two methods viz., Gaussian elimina Gauss-Jordan methods are called direct methods since the g systems of equations yield solution after an amount computation that can be specified in advance. But in iter methods (or indirect methods) we first develop a rule to fin best possible solution. We start with an initial approxi solution and apply the rule to get a better solution. This soluti again subjected to the rule to get a still better solution and so c the rule is applied repetitively, then each successive calculatio determine the next approximation to the solution is calle iteration. The successive approximations themselves are c iterates.

Note : **Iteration method is self-correcting method, si the error made in any computation is corrected in subsequent iterations.**

Gauss - Seidel Iterative Method

Let the given system of equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = C_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = C_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = C_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = C_n$$

Such system is often amenable to an iterative process in which the system is first rewritten in the form

$$x_1 = \frac{1}{a_{11}}(C_1 - a_{12}x_2 - a_{13}x_3 - \ldots\ldots - a_{1n}x_n) \quad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}(C_1 - a_{21}x_1 - a_{23}x_3 - \ldots\ldots - a_{2n}x_n) \quad \ldots (2)$$

$$x_3 = \frac{1}{a_{33}}(C_3 - a_{31}x_1 - a_{32}x_2 - \ldots\ldots - a_{3n}x_n) \quad \ldots (3)$$

2

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n = \frac{1}{a_{nn}}(C_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,\,n-1}\,x_{n-1}) \quad \ldots (4)$$

First let us assume that $x_2 = x_3 = \ldots\ldots\ldots = x_n = 0$ in (1) and find x_1. Let it be $x_1{*}$. Putting $x_1{*}$ for x_1 and $x_3 = x_4 = \ldots = x_n = 0$ in (2) we get the value for x_2 and let it be $x_2{*}$. Putting $x_1{*}$ for x_1 and $x_2{*}$ for x_2 and $x_3 = x_4 = \ldots = x_n = 0$ in (3) we get the value for x_3 and let it be $x_3{*}$. In this way we can find the first approximate values for $x_1, x_2, \ldots\ldots, x_n$. Similarly we can find the better approximate value of $x_1, x_2, \ldots\ldots, x_n$ by using the relation

$$x_1{*} = \frac{1}{a_{11}}(C_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n)$$

$$x_2{*} = \frac{1}{a_{22}}(C_2 - a_{21}x_1{*} - a_{23}x_3 - \ldots\ldots - a_{2n}x_n)$$

Inverse of a Matrix

Gauss - Jordan Method

Let us consider a 3×3 non singular matrix A. If the matrix X is the inverse of A, then $AX = I$

$$\text{i.e.,} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix equation gives nine equations which are equivalent to the three system of equations.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By using Gauss Jordan elimination method we can find the unknowns $x_{11}, x_{21}, x_{31}, \ldots$, etc., which gives the inverse of the given matrix. The method is explained in the following example.

Example

Find the inverse of the matrix $\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ *by Gauss – Jordan method.*

[A.U. Nov. 9...]

Solution

Let $$A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$$

and $$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$ be the inverse of A

So that, $AX = I$

Step 1 : Write the augmented system

$$\left[\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

Step 2 : $R_1 \to R_1 \div 5$

$$\begin{bmatrix} 1 & \frac{-2}{5} & \frac{1}{5} & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

Step 3 : $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & \frac{-2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{26}{5} & \frac{-3}{5} & 1 \end{bmatrix}$$

Step 4 : $R_2 \to R_2 \div \frac{26}{5}$

$$\begin{bmatrix} 1 & \frac{-2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{-3}{26} & \frac{5}{26} \end{bmatrix}$$

Step 5 : $R_1 \to R_1 + \left(\frac{2}{5}\right) R_2$

$$\begin{bmatrix} 1 & 0 & \frac{2}{13} & \frac{1}{13} \\ 0 & 1 & \frac{-3}{26} & \frac{5}{26} \end{bmatrix}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} \frac{2}{13} & \frac{1}{13} \\ \frac{-3}{26} & \frac{5}{26} \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$$

Using Gauss – Jordan method find the inverse of the ma

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$$

[A.U. Apr./May

A 1.13

Solution

Let $$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

A

and $$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$ be the inverse

so that, $$AX = I$$

Step 1 : Write the augmented system.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

Step 2 : $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow 2R_3 - R_1$ 6-2, 10

$$\begin{bmatrix} 2 & 2 & 3 & 1 & \\ 1 & -1 & - & & \\ 0 & & & & \end{bmatrix}$$

Step 3 : $\rightarrow \quad + 4R_2$

$$\begin{bmatrix} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix}$$

Step 4 : $R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix}$$

Step 5 : $R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 2 & 0 & 3 & 19 & -14 & -8 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix}$$

Step 6 : $R_1 \to R_1 + 3R_3$

$$\begin{bmatrix} 2 & 0 & 0 & 4 & -2 & -2 \\ 0 & -1 & 0 & 9 & -7 & -4 \\ 0 & 0 & -1 & -5 & 4 & 2 \end{bmatrix}$$

Step 7 : $R_1 \to R_1 \div 2$, $R_2 \to R_2 \div (-1)$, $R_3 \to R_3 \div (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{bmatrix}$$

A

Hence the inverse of the given matrix is

$$\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$$

Example 5

Find the inverse of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ using Gauss – Jordan method. **[A.U. Nov. '04, Apr '00, Oct. '96]**

Solution

Let
$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

and
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$ be the inverse of A.

so that, $AX = I$

Step 1 : Write the augmented system.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Step 2 : $R_2 \to R_2 - R_1, \quad R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Step 3 : $R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

5

Step 4 : $R_2 \to R_2 \div 2, \quad R_3 \to R_3 \div -4$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right]$$

Step 5 : $R_1 \to R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right]$$

Step 6 : $R_2 \to R_2 + 3R_3$

2

$R_1 - R_1$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Step 7 : $R_1 \to R_1 - 2\,R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Example 4

Newton's Forward Interpolation Formula

We know that

$\Delta y_0 = y_1 - y_0$ i.e., $y_1 = y_0 + \Delta y_0 = (1 + \Delta)\, y_0$

$\Delta y_1 = y_2 - y_1$ i.e., $y_2 = y_1 + \Delta y_1 = (1 + \Delta)\, y_1 = (1 + \Delta)^2\, y_0$

$\Delta y_2 = y_3 - y_2$ i.e., $y_3 = y_2 + \Delta y_2 = (1 + \Delta)\, y_2 = (1 + \Delta)^3\, y_0$

In general, $y_n = (1 + \Delta)^n\, y_0$

Expanding $(1 + \Delta)^n$ by using Binomial theorem we have

$$y_n = \left\{1 + n\Delta + \frac{n\,(n-1)\,\Delta^2}{2!} + \frac{n\,(n-1)\,(n-2)\,\Delta^3}{3!} + \ldots\right\} y_0$$

$$y_n = f\,(x_0 + nh) = y_0 + n\,\Delta y_0 + \frac{n\,(n-1)}{2!}\,\Delta^2 y_0 + \frac{n\,(n-1)\,(n-2)}{3!}\,\Delta^3 y_0 + \ldots$$

This result is known as Gregory–Newton forward interpolation (or) Newton's formula for equal intervals.

Note 1 : This formula is very useful and gives greater accuracy when $x_0 + nh$ is near the beginning of the table.

Note 2 : The first two terms of the above formula will give the linear interpolation, the third term will give the parabolic interpolation and so on.

Example 1

Find the missing y_x values from the first difference provided

y_x	0	–	–	–	–	–
Δy_x	0	1	2	4	7	11

Solution

Let the unknowns be a, b, c, d, e. Then we have the following table.

y_x	0	a	b	c	d	e
Δy_x	0	1	2	4	7	11

By definition, we get,

$$a - 0 = 1 \quad i.e., \quad a = 1$$
$$b - a = 2 \quad \Rightarrow \quad b = 2 + a = 3$$
$$c - b = 4 \quad \Rightarrow \quad c = 4 + b = 7$$
$$d - c = 7 \quad \Rightarrow \quad d = 7 + c = 14$$
$$e - d = 11 \quad \Rightarrow \quad e = 11 + d = 25$$

Example 2

A function $f(x)$ is given by the following table. Find $f(0.2)$ a suitable formula.

x	0	1	2	3	4	5	6
$f(x)$	176	185	194	203	212	220	229

Solution

The difference table is as follows :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
[illegible]	176 (y_0)	Δy_0					
		9	($\Delta^2 y_0$)				
[illegible]	185		0	($\Delta^3 y_0$)			
		9		0	($\Delta^4 y_0$)		
[illegible]	194		0		0	($\Delta^5 y_0$)	
		9		0		-1	($\Delta^6 y_0$)
[illegible]	203		0		-1		5
		9		-1		4	
4	212		-1		3		
		8		2			
[illegible]	220		1				
		9					
6	229						

Here $x_0 = 0$, $h = 1$, $y_0 = 176 = f(x_0)$

We have to find the value of $f(0.2)$. By Newton's forward interpolation formula we have,

$$\boxed{y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \ldots}$$

$$y(0.2) = ?$$

$$x_0 + nh = 0.2$$

i.e., $$0 + n \cdot 1 = 0.2$$

i.e., $$n = 0.2$$

$$\therefore y(0.2) = 176 + (0.2)(9) + \frac{(0.2)(0.2-1)}{2} \cdot 0$$

$$= 176 + 1.8$$

$$= 177.8$$

Hence $\boxed{f(0.2) = 177.8}$

Example

The following table gives the population of a town during the last six census. Estimate, using Newton's interpolation formula, the increase in the population during the period 1946 to 1948.

Year	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	13	20	27	39	52

Solution

27 13 12 27 39 12 13 20

The difference table is as follows

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$1911(x_0)$	$12\ (y_0)$	Δy_0				
		1	$(\Delta^2 y_0)$			
1921	13		6	$(\Delta^3 y_0)$		
		7		-6	$(\Delta^4 y_0)$	
1931	20		0		11	$(\Delta^5 y_0)$
		7		5		-20
1941	27		5		-9	$(\Delta^5 y_0)$ -20
		12		-4		
1951	39		1			
		13				
1961	52					

Here $x_0 = 1911$, $h = 10$, $y_0 = 12$

12 13 20 27 39 52

By Newton's formula we have

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n\,(n-1)}{2!}\Delta^2 y_0 + \frac{n\,(n-1)\,(n-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y\,(1946) = ?$$

i.e., $\quad x_0 + nh = 1946$

i.e., $1911 + n \cdot 10 = 1946$ i.e., $n = 3.5$

1946

$$
\begin{aligned}
\therefore y(1946) &= 12 + (3.5)(1) + \frac{(3.5)(3.5-1)}{2} \times 6 \\
&\quad + \frac{(3.5)(3.5-1)(3.5-2)}{6} \times (-6) \\
&\quad + \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{24} \times 11 \\
&\quad + \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{120} \times (-20) \\
&= 12 + 3.5 + 26.25 - 13.125 + 3.0078 + 0.5469 \\
&= 12 + 3.5 + 26.25 + 3.0078 + 0.5469 - 13.125 \\
&= 32.18
\end{aligned}
$$

∴ The population in the year 1946 is 32.18.

To find the population in the year 1948 :

To find $y(1948)$.

$$
\begin{aligned}
i.e., \quad x_0 + nh &= 1948 \\
\therefore 1911 + n \cdot 10 &= 1948 \\
n &= 3.7
\end{aligned}
$$

$u = \frac{1948 - 1911}{10}$

$$
\begin{aligned}
\therefore y(1948) &= 12 + 3.7 + \frac{(3.7)(3.7-1)}{2} \times 6 \\
&\quad + \frac{3.7(3.7-1)(3.7-2)}{6} \times (-6) \\
&\quad + \frac{3.7(3.7-1)(3.7-2)(3.7-3)}{24} \times 11 \\
&\quad + \frac{3.7(3.7-1)(3.7-2)(3.7-3)(3.7-4)}{120} \times (-20) \\
&= 12 + 3.7 + 29.97 - 16.983 + 5.4487 + 0.5944 \\
&= 34.73
\end{aligned}
$$

The population in the year 1948 is 34.73.

Increase in the population during the period 1946 to 1948.

$$
\begin{aligned}
&= \text{Population in 1948} - \text{Population in 1946} \\
&= 34.73 - 32.18 \\
&= \boxed{2.55 \text{ thousands}}
\end{aligned}
$$