UNIT I:

Elasticity:

Three types of elastic moduli – Poisson's ratio – Bending of beams– Expression for bending moment –Cantilever–Depression of the loaded end of a Cantilever

Expression for Young's modulus (uniform and non–uniform bending) – experimental determination of Young's modulus using pin and microscope method (uniform and non–uniform bending) – Determination of Young's modulus by Koenig's method for non–uniform bending

Torsion of a body – expression for couple per unit twist – determination of rigidity modulus – Static torsion method with scale and telescope – determination of rigidity modulus by torsion pendulum with mass

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Elasticity

CHAPTER

1.1. INTRODUCTION

A body can be deformed (*i.e.*, changed in shape or size) by the suitable application of external forces on it. A body is said to be perfectly elastic, if it regains its original shape or size, when the applied forces are removed. This property of a body to regain its original state or condition on removal of the applied forces is called *elasticity.* A body which does not tend to regain its original shape or size, even when the applied forces are removed, is called a perfectly plastic body. No body, in nature, is either perfectly elastic or perfectly plastic. Quartz fibre is the nearest approach to a perfectly elastic body.

Stress : When an external force is applied on a body, there will be relative displacement of the particles and due to the property of elasticity, the particles tend to regain their original positions. *Stress is defined as the restoring force per unit area.* If a force F is applied normally to the area of cross-section A of a wire, then stress = F/A. Its dimensions are $ML^{-1}T^{-2}$.

Thermal Stresses : Suppose the ends of a rod are rigidly fixed, so as to prevent expansion or contraction. If the temperature of the rod is changed, tensile or compressive stresses, called thermal stresses, will be set up in the rod. If these stresses are very large, the rod may be stressed even beyond its breaking strength. The stress is *tensile* when there is an increase in length. The stress is *compressive* when there is a decrease in length. A tangential stress tries to slide each layer of the body over the layer immediately below it.

Strain : When a deforming force is applied, there is a change in length, shape or volume of the body. The ratio of the change in any dimension to its original value is called strain. It is of three types :-

(1) The ratio of change in length (l) to original length (L) is called *longitudinal strain* (l/L).

(2) Let $ABCD$ be a body with the side CD fixed (Fig. 1.1). Suppose a tangential force F is applied on the upper face AB. The shape of the body is changed to $A'B'CD$. The body is sheared by an angle ϕ. This angle ϕ measured in radians is called the *shearing strain* (ϕ).

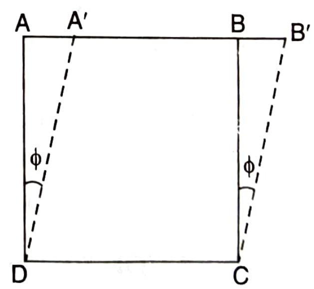


Fig. 1.1

(3) *Volume strain* (*Bulk strain*) : The ratio of change in volume (v) to original volume (V) is called *volume strain* (v/V).

Hooke's Law : *Within elastic limit, the stress is directly proportional to strain.* Stress $\propto$ strain or stress/strain = E . E is a constant called modulus of elasticity.

The dimensional formula of modulus of elasticity is $ML^{-1}T^{-2}$. Its units are Nm^{-2}.

1.2. DIFFERENT MODULI OF ELASTICITY

(1) **Young's modulus (E) :** *It is defined as the ratio of longitudinal stress to longitudinal strain within elastic limits.* Let a wire of length L and area of cross-section A undergo an increase in length l when a stretching force F is applied in the direction of its length.

Then, longitudinal stress = F/A and longitudinal strain = l/L.

$$\therefore \qquad E = \frac{F/A}{l/L} = \frac{FL}{Al}.$$

(2) **Rigidity modulus (G) :** *It is defined as the ratio of tangential stress to shearing strain.*

Consider a solid cube $ABCDEFGH$ (Fig. 1.2). The lower face $CDGH$ is fixed and a tangential force F is applied over the upper face $ABEF$. The result is that each horizontal layer of the cube is displaced, the displacement being proportional to its distance from the fixed plane. Point A is shifted to A', B to B', E to E' and F to F' through an angle ϕ, where $AA' = EE' = l$.

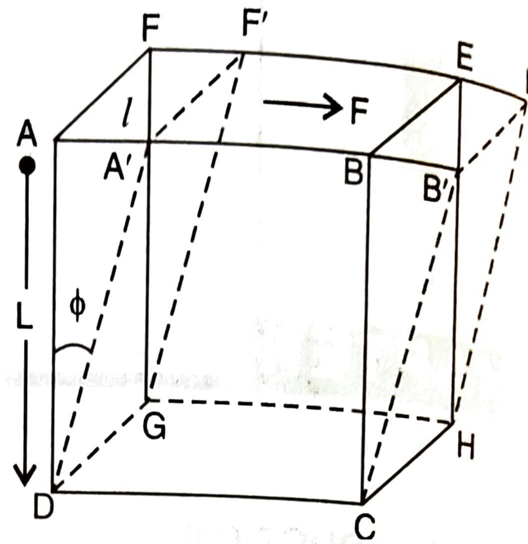


Fig. 1.2

Clearly $\phi = l/L$ where l is the relative displacement of the upper face of the cube with respect to the lower fixed face, distant L from it.

This angle ϕ through which a line originally perpendicular to the fixed face is turned, is a measure of the shearing strain.

$$\text{Now, Rigidity modulus } (G) = \frac{\text{Tangential stress}}{\text{Shearing strain}} = \frac{F/A}{\phi}$$

Here, $\qquad A = L^2$ = Area of face $ABEF$.

$\therefore \qquad G = T/\phi$ where T = Tangential stress.

(3) **Bulk Modulus (K):** *It is defined as the ratio of volume stress (Bulk Stress) to the volume strain.*

When three equal stresses (F/A) act on a body in mutually perpendicular directions, such that there is a change of volume v in its original volume V, we have, Stress = pressure $P = F/A$. Volume strain = $-v/V$. The negative sign indicates that if pressure increases, volume decreases.

$$\therefore \qquad K = \frac{\text{Bulk strees}}{\text{Volume strain}} = \frac{F/A}{-v/V} = \frac{P}{-v/V}$$

Poisson's Ratio (ν): When a wire is stretched, it becomes longer but thinner, *i.e.*, although its length increases, its diameter decreases. When a wire elongates freely in the direction of a tensile stress, it contracts laterally (*i.e.*, in a direction perpendicular to the force). *The ratio of lateral contraction to the longitudinal elongation is called Poisson's ratio.* It is denoted by the letter ν.

If unit stress causes an extensional strain λ in its own direction and lateral contractional strain μ in a perpendicular direction, $\nu = \mu/\lambda$.

1.14. DEFINITIONS

2m **Beam :** *A beam is defined as a rod or bar of uniform cross-section (circular or rectangular) whose length is very much greater than its thickness.*

Bending Couple : If a beam is fixed at one end and loaded at the other end, it bends. The load acting vertically downwards at its free end and the reaction at the support acting vertically upwards constitute the bending couple. This couple tends to bend the beam clockwise. Since there is no rotation of the beam, the external bending couple must be balanced by another equal and opposite couple which comes into play inside the body due to the elastic nature of the body. The moment of this elastic couple is called the internal bending moment. When the beam is in equilibrium,

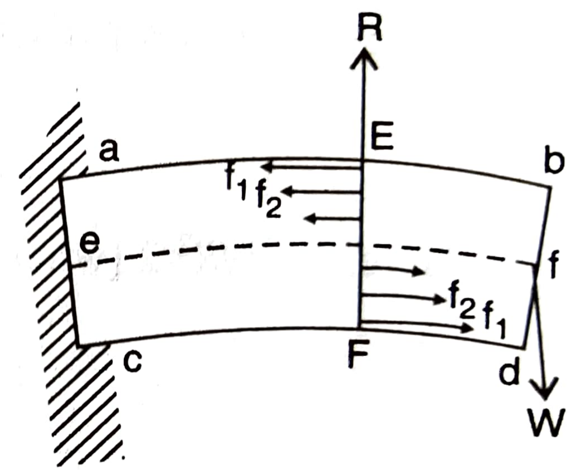


Fig. 1.12

the external bending moment = the internal bending moment.

Plane of Bending : The plane of bending is the plane in which the bending takes place and the bending couple acts in this plane. In Fig. 1.12, the plane of paper is the plane of bending.

Neutral Axis : When a beam is bent as in Fig. 1.12, filaments like *ab* in the upper part of the beam are elongated and filaments like *cd* in the lower part are compressed. Therefore, there must be a filament like ef in between, which is neither elongated nor compressed. Such a filament is known as the neutral filament and the axis of the beam lying on the neutral filament is the neutral axis. The change in length of any filament is proportional to the distance of the filament from the neutral axis.

1.15. EXPRESSION FOR THE BENDING MOMENT

Consider a portion of the beam to be bent into a circular arc, as shown in Fig. 1.13. ef is the neutral axis. Let R be the radius of curvature of the neutral axis and θ the angle subtended by it at its centre of curvature C.

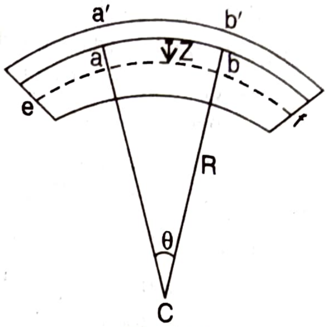


Fig. 1.13

Filaments above ef are elongated while filaments below ef are compressed. The filament ef remains unchanged in length.

Let $a'b'$ be a filament at a distance z from the neutral axis. The length of this filament $a'b'$ before bending is equal to that of the corresponding filament on the neutral axis ab.

We have, original length $= ab = R\theta$.

Its extended length $= a'b' = (R + z)\,\theta$

Increase in its length $= a'b' - ab = (R + z)\,\theta - R\theta = z\,.\,\theta$.

$$\therefore \qquad \text{Linear strain} = \frac{\text{increase in length}}{\text{original length}} = \frac{z\,.\,\theta}{R\,.\,\theta} = \frac{z}{R}$$

If E is the Young's modulus of the material,

$$E = \text{Stress/Linear strain}$$

i.e.,

$$\text{Stress} = E \times \text{Linear strain} = E\,(z/R)$$

If δA is the area of cross-section of the filament,

the tensile force on the area δA = stress × area $= \dfrac{E\,.\,z}{R}\,\delta A$.

Moment of this force about the neutral axis ef

$$= \frac{E\,.\,z}{R}\,\delta A\,.\,z = \frac{E}{R}\,\delta A\,.\,z^2.$$

$$\left.\begin{array}{l}\text{The sum of the moments of}\\ \text{forces acting on all the filaments}\end{array}\right\} = \Sigma\,\frac{E}{R}\,\delta A\,.\,z^2$$

$$= \frac{E}{R}\,\Sigma\,\delta A\,.\,z^2$$

$\Sigma\delta A\,.\,z^2$ is called the geometrical moment of inertia of the cross-section of the beam about an axis through its centre perpendicular to the plane of bending. It is written as equal to Ak^2. *i.e.*, $\Sigma\delta A\,.\,z^2 = Ak^2$. ($A$ = Area of cross-section and k = radius of gyration).

But the sum of moments of forces acting on all the filaments is the internal bending moment which comes into play due to elasticity.

Thus, bending moment of a beam $= EAk^2/R$.

Notes : (*i*) For a rectangular beam of breadth b, and depth (thickness) d, $A = bd$ and $k^2 = d^2/12$.

$$\therefore \qquad Ak^2 = bd^3/12.$$

(*ii*) For a beam of circular cross-section of radius r, $A = \pi\, r^2$ and $k^2 = r^2/4$.

$$\therefore \qquad Ak^2 = \pi r^4/4.$$

(*iii*) EAk^2 is called the flexural rigidity of the beam.

1.16. DEPRESSION OF THE LOADED END OF A CANTILEVER

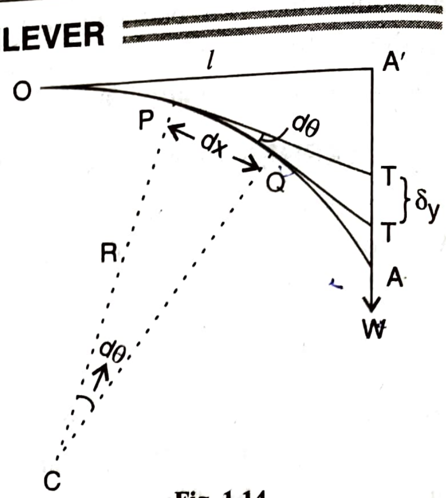


Fig. 1.14

Cantilever : A cantilever is a beam fixed horizontally at one end and loaded at the other end.

Let OA be a cantilever of length l fixed at O and loaded with a weight W at the other end. OA' is the unstrained position of the beam. Let the depression $A'A$ of the free end be y (Fig. 1.14). Let us consider an element PQ of the beam of length dx at a distance ($QA = x$) from the loaded end. C is the centre of curvature of the element PQ and R its radius of curvature. The load W at A and the force of reaction W at Q constitute the external couple, so that, the external bending moment = $W.x$.

The internal bending moment = EAk^2/R .

For equilibrium, $Wx = EA\,k^2/R$ or $R = EA\,k^2/Wx$ ...(1)

Draw tangents at P and Q meeting the vertical line at T and S respectively. Let $TS = dy$ and $d\theta$ = Angle between the tangents. Then, $\angle PCQ$ also = $d\theta$.

Now, $$PQ = dx = R\,d\theta \text{ or } d\theta = \frac{dx}{R} = dx \cdot \frac{Wx}{E\,A\,k^2}$$ (From Eq. 1)

We have, $$dy = x\,d\theta = x\,.\,\frac{Wx\,dx}{EAk^2} = \frac{Wx^2\,dx}{E\,A\,k^2}$$ ...(2)

$$\therefore \left.\begin{array}{l}\text{the total depression of}\\ \text{the end of the cantilever}\end{array}\right\} = y = \int_0^l \frac{Wx^2}{E\,A\,k^2}\,dx = \frac{W\,l^3}{3E\,A\,k^2}$$

Angle between the tangents at the ends of a cantilever :

Since the beam is fixed horizontally at O, the tangent at O is horizontal. If a tangent is drawn at A (the free end of the bent bar), it makes an angle θ with the horizontal.

$$\left.\begin{array}{l}\text{Angle between the}\\ \text{tangents at } P \text{ and } Q\end{array}\right\} = d\theta = \frac{Wx}{EA\,k^2}\,dx.$$

$$\left.\begin{array}{l}\text{Angle between the}\\ \text{tangents at } O \text{ and } A\end{array}\right\} = \theta = \int_0^l = \frac{Wx}{E\,A\,k^2}\,dx$$

$$\therefore \qquad \theta = \frac{W\,l^2}{2E\,A\,k^2}.$$

Work done in uniform bending. Consider a beam bent uniformly by an external couple. Let A be the area of cross-section of the beam. Consider a filament of area of cross-section δA at a distance z from the neutral axis (Fig. 1.13). Then,

the tensile force on the area $\delta A = \dfrac{Ez}{R}\,\delta A.$

The linear strain of this filament = z/R. If l is the length of the filament, then, the extension of the filament = zl/R.

$$\left.\begin{array}{l}\text{The work done in}\\ \text{bending the filament}\end{array}\right\} = \frac{1}{2}\text{ force} \times \text{extension}$$

$$= \frac{1}{2}\frac{Ez}{R}\,\delta A \times \frac{zl}{R} = \frac{1}{2}\frac{El}{R^2} \times z^2\,.\,\delta A.$$

For uniform bending R is constant. Hence, the work done in bending the whole beam is

$$W = \frac{1}{2}\frac{El}{R^2}\Sigma z^2\,\delta A = \frac{1}{2}\frac{El}{R^2}\times Ak^2 = \frac{1}{2}\frac{EAk^2}{R}\times\frac{l}{R}.$$

Here, EAk^2/R = the bending moment and l/R = the angle subtended by the bent beam at its centre of curvature.

$\therefore$ The work done in uniform bending $= \frac{1}{2}$ (bending moment) $\times$ (Angle subtended by the bent beam at its centre of curvature).

Example 13 : *Obtain an expression for the depression at the free end of a heavy beam clamped horizontally at one end and loaded at the other end.*

Consider an element PQ of the beam of length dx at a distance x from the fixed end O (Fig. 1.14). Now, in addition to the load W acting at A, a weight equal to that of the portion $(l-x)$ of the beam also acts at its mid-point. Let W_1 be the weight of the beam. Then, the weight per unit length of the beam $= W_1/l$. Now, we have an additional weight $W_1(l-x)/l$ acting at a distance $(l-x)/2$ from Q. Therefore,

$$\left.\begin{array}{l}\text{total moment of the}\\ \text{external couple applied}\end{array}\right\} = W(l-x) + \frac{W_1}{l}(l-x)\frac{(l-x)}{2}$$

$$= W(l-x) + \frac{W_1}{2l}(l-x)^2$$

The beam being in equilibrium, this must be balanced by the bending moment EAk^2/R. Therefore,

$$W(l-x) + \frac{W_1}{2l}(l^2 - 2l\,.\,x + x^2) = \frac{EAk^2}{R} = EAk^2\left(\frac{d^2y}{dx^2}\right)$$

Integrating,

$$W\left(lx - \frac{x^2}{2}\right) + \frac{W_1}{2l}\left(l^2\,.\,x - l\,.\,x^2 + \frac{x^3}{3}\right) = EAk^2\frac{dy}{dx} + C$$

where C is a constant of integration.

Since at $x = 0$, $dy/dx = 0$, we have $C = 0$.

Integrating once again,

$$EAk^2\int_0^y dy = W\int_0^l (lx - x^2/2)\,dx + \frac{W_1}{2l}\int_0^1 (l^2x - lx^2 + x^3/3)\,dx$$

or $$EAk^2y = W\left(\frac{l^3}{3}\right) + \frac{W_1}{2l}\left(\frac{l^4}{4}\right)$$

or $$EAk^2y = \frac{Wl^3}{3} + \frac{W_1l^3}{8},$$

or $$y = \left(W + \frac{3}{8}W_1\right)\frac{l^3}{3EAk^2}$$

$8\,E\,A\,k^2$

1.21. MEASUREMENT OF YOUNG'S MODULUS—BY BENDING OF A BEAM

(1) Non-uniform Bending : The given beam is symmetrically supported on two knife-edges (Fig. 1.21). A weight-hanger is suspended by means of a loop of thread from the point C exactly midway between the knife-edges. A pin is fixed vertically at C by some wax. A travelling microscope is focussed on the tip of the pin such that the horizontal cross-wire coincides with the tip of the pin. The reading in the vertical traverse scale of microscope is noted. Weights are added in equal steps of m kg and the corresponding readings are noted. Similarly, readings are noted while unloading. The results are tabulated as follows :

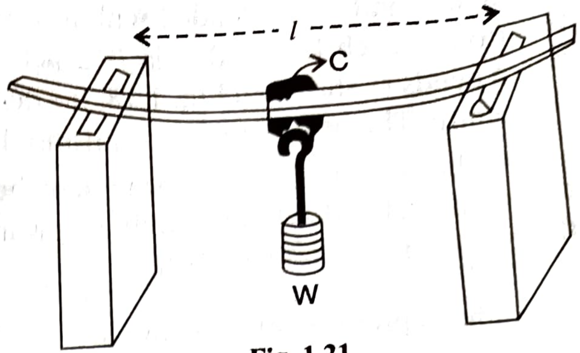


Fig. 1.21

Load in kg	*Readings of the microscope*			*y for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The mean depression y is found for a load of M kg. The length of the beam (l) between the knife-edges is measured. The breadth b and the thickness d of the beam are measured with a vernier calipers and screw gauge respectively.

Then, $$y = \frac{Wl^3}{48\,EA\,k^2} \text{ or } E = \frac{Wl^3}{48\,Ak^2 y}$$

or $$E = \frac{Mg\,l^3}{48 \times (bd^3/12) \times y} \qquad (\because W = Mg \text{ and } Ak^2 = bd^3/12)$$

$\therefore$ $$E = \frac{Mgl^3}{4\,bd^3 y}$$

Example 15 : *In an experiment a rod of diameter 0.0126 m was supported on two knife-edges, placed 0.7 metre apart. On applying a load of 0.9 kg exactly midway between the knife-edges, the depression on the middle point was observed to be 0.00025 m. Calculate the Young's modulus of the substance.*

$$E = \frac{Mgl^3}{12 y\pi r^4} = \frac{(0.9)(9.8)\,(0.7)^3}{12\,(0.00025)\,\pi\,(0.0063)^4}$$

$\therefore$ $$E = 2.039 \times 10^{11}\ \text{Nm}^{-2}$$

(2) Uniform bending : The given beam is supported symmetrically on two knife-edges A and B (Fig. 1.22). Two equal weight-hangers are suspended, so that their distances from the kinfe-edges are equal. The elevations of the centre of the beam may be measured accurately by using a single optic level (L). The front leg of the single optic lever rests on the centre of the loaded beam and the hind legs are supported on a separate stand. A vertical scale (S) and telescope (T) are arranged in front of the mirror. The telescope is focussed on the mirror and adjusted so that the reflected image of the scale in the mirror is seen through the telescope. The load on each hanger is increased in equal steps of m kg and the corresponding readings on the scale are noted. Similarly, readings are noted while unloading. The results are tabulated as follows :

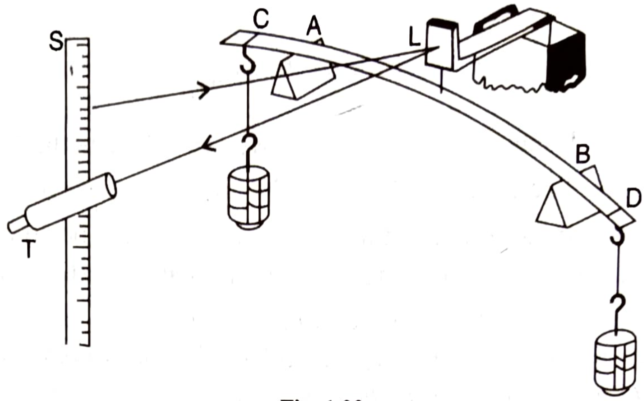


Fig. 1.22

Load in kg	*Readings of the scale as seen in the telescope*			*Shift in reading for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The shift in scale reading for M kg is found from the table. Let it be S. If

D = The distance between the scale and the mirror,

x = the distance between the front leg and the plane containing the two hind legs of the optic lever,

then $$y = Sx/2D.$$

The length of the beam l between the knife-edges, and a, the distance between the point of suspension of the load and the nearer knife-edge ($AC = BD = a$) are measured. The breadth b and the thickness d of the beam are also measured.

Then, $$y = \frac{Wal^2}{8\,E\,A\,k^2} \text{ or } \frac{Sx}{2D} = \frac{Mgal^2}{8\,E\,(b\,d^3/12)}$$

[Since $W = Mg$ and $Ak^2 = b\,d^3/12$]

$$\therefore \quad E = \frac{3\,Mgal^2 D}{Sxbd^3}$$

Pin and Microscope Method : The given beam is supported symmetrically on two knife-edges A and B. Two equal weight-hangers are suspended so that their distances from the knife-edges are equal. A pin is placed vertically at the centre of the beam. The tip of the pin is viewed by a microscope. The load on each hanger is increased in equal steps of m kg and the corresponding microscope readings are noted. Similarly, readings are noted while unloading. The results are tabulated as follows :

Load in kg	*Readings of the microscope*			*y for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The mean elevation (y) of the centre for M kg is found. The length of the beam l between the knife-edges and a, the distance between the point of suspension of the load and the nearer knife-edge ($AC = BD = a$) are measured. The breadth b and the thickness d of the beam are also measured.

$$y = \frac{Wal^2}{8EAk^2} = \frac{Mgal^2}{8E(bd^3/12)} \qquad \left(\because\ W = Mg \text{ and } Ak^2 = \frac{bd^3}{12}\right)$$

$$\therefore \qquad E = \frac{3Mgal^2}{2bd^3 y}$$

Using the above formula, we can calculate the Young's modulus of the material of the beam.

Example 16 : *Distinguish between uniform and non-uniform bending.*

In uniform bending every element of the beam is bent with the same radius of curvature (R). In non-uniform bending, R is not the same for all the elements in the beam.

Example 17 : *Determine the Young's modulus of the material of a rod, if it is bent uniformly over two knife-edges separated by a distance of 0.6 m and loads of 2.5 kg are hung at 0.18 m away from the knife-edges. The breadth and thickness of the rod are 0.025 m and 0.005 m respectively. The elevation at the middle of the rod is 0.007 m.*

$$E = \frac{3Mgal^2}{2bd^3 y} = \frac{3 \times 2.5 \times 9.8 \times 0.18 \times (0.6)^2}{2 \times 0.025 \times (0.005)^3 \times 0.007}$$

$$= 1.088 \times 10^{11}\ \text{Nm}^{-2}.$$

1.23. KÖNIG'S METHOD

The beam is supported on two knife-edges K_1 and K_2 separated by a distance l. Two plane mirrors m_1 and m_2 are fixed near the two ends of the beam at equal distances beyond the knife-edges. [Fig. 1.24 (*a*)]. The two plane mirrors face each other and they are inclined slightly outwards from the vertical.

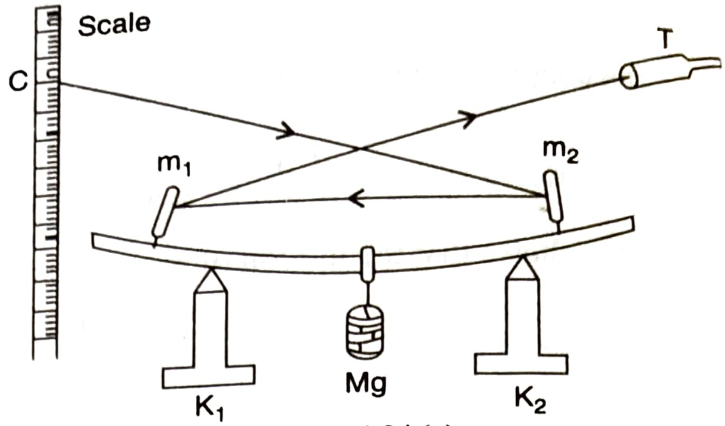


Fig. 1.24 (*a*)

An illuminated translucent scale and a telescope (T) are arranged as shown. The reading of a point C on the scale as reflected first by m_2 and then by m_1 is viewed in the telescope. Let the load suspended at the mid-point of the beam be M. The beam is then bent and the bending is non-uniform. The mirrors at the ends are turned towards each other [Fig. 1.24 (*b*)]. Let the shift in the scale reading be s. The Young's modulus of the material of the beam is then calculated from the relation

$$E = \frac{3Mgl^2\,(2D+L)}{2bd^3 s}$$

where l = Distance between the knife-edges

D = Distance between the scale and the remote mirror, m_2

L = Distance between the two mirrors.

s = Shift in scale reading for a load of M kg

b = Breadth of the beam

d = Thickness of the beam

The formula can be deduced as explained below.

Let θ be the angle through which each end of the beam has been turned due to loading. Then,

$$\theta = \frac{Wl^2}{16\,EAk^2}$$

The mirrors m_1 and m_2 also turn through the same angle θ due to loading. In Fig. 1.24(*b*), m_1 and m_2 represent the initial and m_1' and m_2' the displaced positions of the mirrors. Originally, the image of the scale division at C coincides with the cross-wire and finally when the load is applied, H is seen to be in coincidence with the cross-wire. For convenience in evaluating θ, consider the rays of light to be reversed in their path.

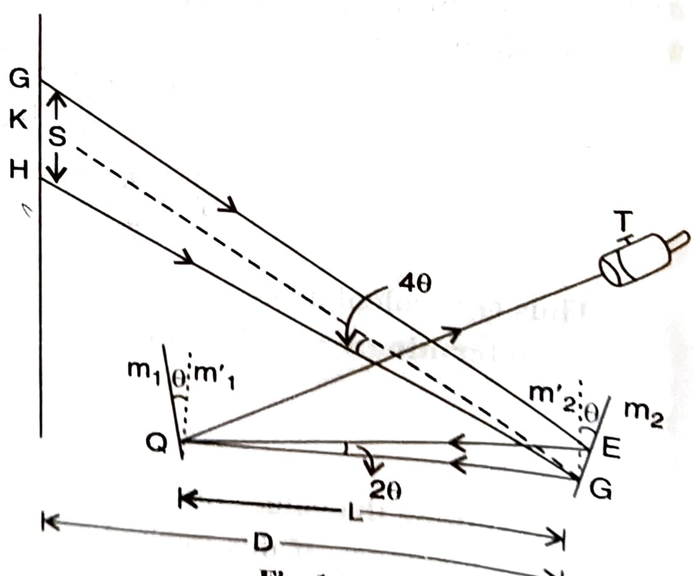


Fig. 1.24 (b)

$TQEC$ will be the original path. When m_1 is turned through an angle θ to the position m_1', QE is turned through 2θ and strikes m_2 at G. Then $EG = L2\theta$. The ray GH is turned through an angle 4θ, since, in addition to QE having moved through 2θ, m_2 itself has turned through θ. Draw GK parallel to EC. Then, $\angle KGH = 4\theta$ and $CK = EG$. $KH = D\,4\theta$

$\therefore$ The total shift in scale reading $= s = CK + KH$

$$= EG + KH \qquad (\because \; CK = EG)$$

$$= L2\theta + D\,4\theta$$

$$= (L + 2D)\,2\,\theta$$

But
$$\theta = \frac{Wl^2}{16\,EAk^2}$$

Hence,
$$s = (L + 2D) \times 2 \times \frac{Wl^2}{16\,EAk^2}$$

$\therefore$
$$E = \frac{Wl^2(L + 2D)}{8\,Ak^2 s}$$

Now $Ak^2 = bd^3/12$ for a beam of rectangular cross-section and

$$W = Mg.$$

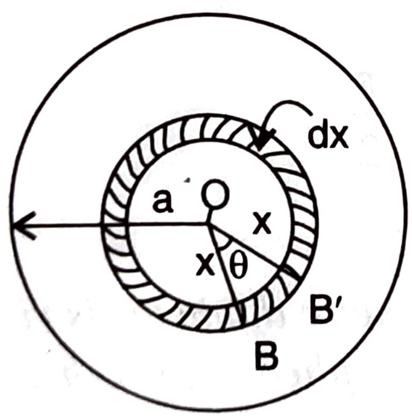
$\therefore$
$$E = \frac{Mgl^2\;(L + 2D)}{8\,(bd^3\,/12)s} = \frac{3Mgl^2\;(2D + L)}{2bd^3 s}$$

1.9. TORSION OF A BODY

When a body is fixed at one end and twisted about its axis by means of a torque at the other end, the body is said to be under torsion. Torsion involves shearing strain and so the modulus involved is the rigidity modulus.

Torsion of a cylinder–Expression for torque per unit Twist

Consider a cylindrical wire of length L and radius a fixed at its upper end and twisted through an angle θ by applying a torque at the lower end. Consider the cylinder to consist of an infinite number of hollow co-axial cylinders. Consider one such cylinder of radius x and thickness dx [Fig. 1.7(i)].



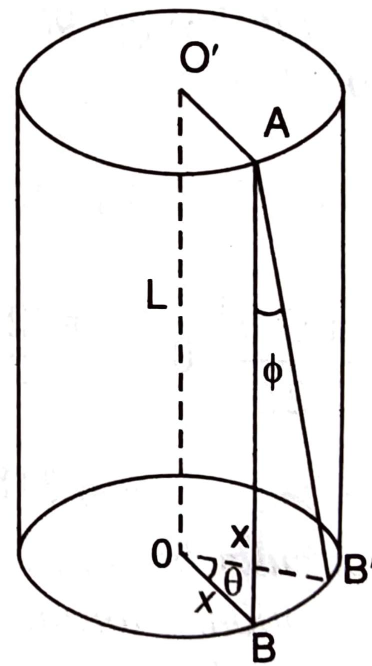


Fig. 1.7

A line such as AB initially parallel to the axis OO' of the cylinder is displaced to the position AB' through an angle ϕ due to the twisting torque [Fig. 1.7(*ii*)]. The result of twisting the cylinder is a shear strain. The angle of shear $= \angle BAB' = \phi$.

Now $\quad BB' = x.\theta = L\phi$ or $\phi = x.\theta/L$

We have, rigidity modulus $= G = \dfrac{\text{Shearing stress}}{\text{Angle of shear } (\phi)}$

$\therefore \quad$ shearing stress $= G \,.\, \phi = Gx\, \theta/L$

But, $\quad$ shearing stress $= \dfrac{\text{Shearing force}}{\text{Area on which the force acts}}$

$\therefore \quad$ Shearing force = Shearing stress × Area on which the force acts.

The area over which the shearing force acts $= 2\pi\, x\, dx$

Hence, the shearing force $= F = \dfrac{G\,x\,\theta}{L} \times 2\pi x\, dx$

$$\left.\begin{array}{r}\text{The moment of this}\\ \text{force about the axis}\\ OO' \text{ of the cylinder}\end{array}\right\} = \frac{G\,x\,\theta}{L} 2\pi\, x\, dx.x = \frac{2\pi G\theta}{L} x^3\, dx$$

$$\therefore \quad \left.\begin{array}{r}\text{Twisting torque on}\\ \text{the whole cylinder}\end{array}\right\} = C = \int_0^a \frac{2\,\pi\, G\,\theta}{L} x^3\, dx$$

or $$C = \frac{\pi\, G\, a^4\, \theta}{2L}$$

$$\left.\begin{array}{r}\text{The torque per unit twist (\textit{i.e.},}\\ \text{the torque when } \theta = 1 \text{ radian)}\end{array}\right\} = c = \frac{\pi\, G\, a^4}{2L}$$

Note 1: When an external torque is applied on the cylinder to twist it, at once an internal torque, due to elastic forces, comes into play. In the equilibrium position, these two torques will be equal and opposite.

Note 2: If the material is in the form of a hollow cylinder of internal radius a and external radius b, then,

$$\left.\begin{array}{r}\text{The torque acting}\\ \text{on the cylinder}\end{array}\right\} = C = \int_a^b \frac{2\,\pi\, G\,\theta}{L} x^3\, dx = \frac{\pi\, G\,\theta}{2L}(b^4 - a^4)$$

Torque per unit twist $= c = \pi G\,(b^4 - a^4)/(2L)$

***Example 5:** What torque must be applied to a wire one metre long, 10^{-3} metre in diameter in order to twist one end of it through 90°, the other end remaining fixed ? The rigidity of the material of the wire is 2.8×10^{10} Nm^{-2}.*

Here, $L = 1$ m ; $G = 2.8 \times 10^{10}$ Nm^{-2} ; $a = \dfrac{10^{-3}}{2}$ m $= 0.5 \times 10^{-3}$ m ; $\theta = 90° = \pi/2$ radians ;

$$\therefore \quad C = \frac{\pi\, G\, a^4}{2L}\theta = \frac{\pi\,(2.8 \times 10^{10})(0.5 \times 10^{-3})^4}{2 \times 1} \times \frac{\pi}{2}$$

$$= 4.318 \times 10^{-3} \text{ Nm}$$

1.10. DETERMINATION OF RIGIDITY MODULUS—STATIC TORSION METHOD

Searle's apparatus : The experimental rod is rigidly fixed at one end A and fitted into the axle of a wheel W at the other end B (Fig. 1.8). The wheel is provided with a grooved edge over which passes a tape. The tape carries a weight hanger at its free end. The rod can be twisted by adding weights to the hanger. The angle of twist can be measured by means of two pointers fixed at Q and R which move over circular scales S_1 and S_2. The scales are marked in degrees with centre zero.

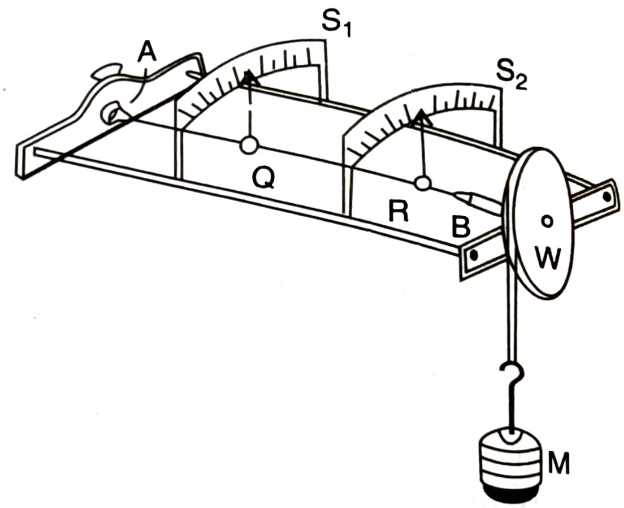


Fig. 1.8

With no weights on the hanger, the initial readings of the pointers on the scales are adjusted to be zero. Loads are added in steps of m kg (conveniently 0.2 kg). The readings on the two scales are noted for every load, both while loading and unloading. The experiment is repeated after reversing the twisting torque by winding the tape over the wheel in the opposite way. The observations are tabulated.

The readings in the last column give the twist for a load of M kg for the length QR (= L) of the rod.

The radius a of the rod and the radius R of the wheel are measured.

If a load of M kg is suspended from the free end of the tape, the twisting torque = MgR.

The angle of twist = θ degrees = $\theta . \pi/180$ radians.

$$\therefore \quad \text{The restoring torque} = \frac{\pi G a^4}{2L} \cdot \frac{\theta \pi}{180}.$$

$$\text{For equilibrium,} \quad MgR = \frac{\pi G a^4}{2L} \frac{\theta . \pi}{180} \quad \text{or} \quad G = \frac{360 \, M g R L}{\pi^2 a^4 \theta}$$

Since a occurs in the fourth power in the relation used, it should be measured very accurately.

Notes: (1) We eliminate the error due to the eccentricity of the wheel by applying the torque in both clockwise and anticlockwise directions.

(2) We eliminate errors due to any slipping at the clamped end by observing readings at two points on the rod.

1.11. DETERMINATION OF RIGIDITY MODULUS—STATIC TORSION METHOD. (SEARLE'S APPARATUS—SCALE AND TELESCOPE)

A plane mirror strip is fixed to the rod at a distance L from the fixed end of the rod [Fig. 1.9]. A vertical scale (S) and telescope (T) are arranged in front of the mirror. The telescope is focussed

Load	Reading on S_1					Reading on S_2					$\theta = \theta_2 - \theta_1$	θ for M kg
	Torque Clockwise		Torque Anticlockwise		Mean θ_1	Torque Clockwise		Torque Anticlockwise		Mean θ_2		
	Load increasing	Load decreasing	Load increasing	Load decreasing		Load increasing	Load decreasing	Load increasing	Load decreasing			

on the mirror and adjusted so that the reflected image of the scale in the mirror is seen through the telescope. With some dead load W on the weight-hanger, the reading of the scale division coinciding with the horizontal cross-wire is taken. Weights are added in steps of m kg and the corresponding scale readings are taken. Weights are then decreased continuously in steps of m kg and the readings taken again. The torque is reversed now, by passing the tape anticlockwise on the wheel. The readings are taken as before. From these readings, the shift in scale reading (s) for a load m kg is found.

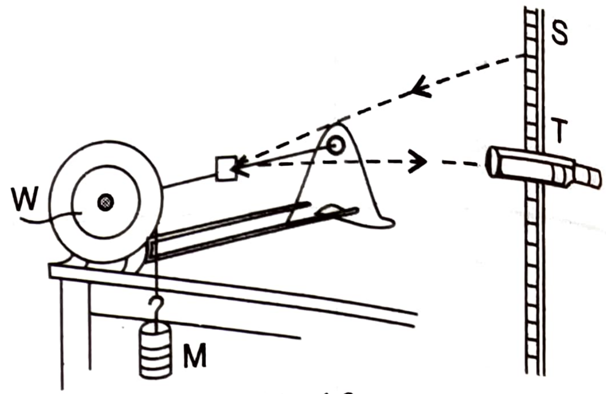


Fig. 1.9

The length L of the rod from the fixed end to the mirror is measured. The mean radius a of the rod is accurately measured with a screw gauge. The radius (R) of the wheel is found by measuring its circumference with a thread. The distance (D) between the scale and the mirror is measured with a metre scale.

G is calculated using the formula $G = \dfrac{4\,mg\,R\,L\,D}{\pi a^4 s}$

Load in kg	*Telescope Reading*						$\frac{X \sim Y}{2}$	*Shift in scale reading for 4m kg*
	Torque clockwise			*Torque anticlockwise*				
	Loading	*Unloading*	*Mean (X)*	*Loading*	*Unloading*	*Mean (Y)*		
W								
W + m								
W + 2 m								
W + 3 m								
W + 4 m								
W + 5 m								
W + 6 m								
W + 7 m								