UNIT- 3

SERIES OF REAL NUMBERS

3.1. convergence and divergence:

3.1A Definition: Series:

The finite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $\langle \{a_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} \rangle$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and

2 mark (E)

$$S_n = a_1 + a_2 + \cdots + a_n \quad (n \in I)$$

The number a_n is called the n^{th} term of the series.

The number S_n is called the n^{th} partial sum of the series.

3.1B. Definition: Convergent Series:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partial sums $S_n = a_1 + \cdots + a_n$ $(n \in I)$.
If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $A \in R$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges to A.

Divergent Series:

If $\{S_n\}_{n=1}^{\infty}$ direoges, we say that $\sum_{n=1}^{\infty} a_n$ diverges.

3.1 C Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to A+B. Also if $c \in R$, then $\sum_{n=1}^{\infty} c a_n$ converges to cA.

Proof:

If $S_n = a_1 + a_2 + \cdots + a_n$ and

$t_n = b_1 + b_2 + \cdots + b_n$,

By hypothesis

$$\lim_{n \to \infty} S_n = A \;\&$$

$$\lim_{n \to \infty} t_n = B.$$

But the n^{th} partial sum of $\sum_{n=1}^{\infty} (a_n + b_n)$ is

$$(a_1 + b_1) + \cdots + (a_n + b_n) = S_n + t_n$$

By Known theorem

If $\lim_{n \to \infty} S_n = L$ and if $\lim_{n \to \infty} t_n = M$, then

$$\lim_{n \to \infty} (S_n + t_n) = L + M.$$

$\therefore$ $S_n + t_n$ approaches $A + B$ as $n \to \infty$.

this proves $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$.

b) If $c \in R$ then $\{cS_n\}_{n=1}^{\infty}$ converges to cA.

(ie) $\lim_{n \to \infty} cS_n = cA$

where $cS_n = c \cdot (a_1 + a_2 + \cdots + a_n)$

$= ca_1 + ca_2 + \cdots + ca_n$

Hence $\sum_{n=1}^{\infty} Ca_n$ converges to CA.

Since the sequence $\{CS_n\}_{n=1}^{\infty}$ converges to CA.

Note:

If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B.

Then $\sum_{n=1}^{\infty} (a_n - b_n)$ converges to $A - B$.

We have

$\sum_{n=1}^{\infty} a_n = A$ then $\sum_{n=1}^{\infty} Ca_n = CA$ in this

Let us take $C = -1$ for the series $\sum_{n=1}^{\infty} b_n$.

$\therefore$ We have $\sum_{n=1}^{\infty} -1 \cdot b_n = -1 \cdot B$

(ie) $\sum_{n=1}^{\infty} \{-b_n\} = -B$.

$\therefore$ From theorem known we have

$$\sum_{n=1}^{\infty} a_n + (-b_n) = A + (-B)$$

(ie) $\sum_{n=1}^{\infty} a_n - b_n = A - B$.

Theorem 3.1 D

If $\sum_{n=1}^{\infty} a_n$ is a convergent series then $\lim_{n\to\infty} a_n = 0$.

Proof:

Given $\sum_{n=1}^{\infty} a_n$ is a convergent series.

We have $\sum_{n=1}^{\infty} a_n = A$.

Then $\lim_{n\to\infty} S_n = A$.

Let $S_n = a_1 + a_2 + \cdots + a_n$

Then $\{S_n\}_{n=1}^{\infty}$ converges to A.

$\Rightarrow \lim_{n\to\infty} S_n = A$, $\lim_{n\to\infty} S_{n-1} = A$.

Now consider

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= A - A$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

Note:

This is the necessary condition for a series to be convergent. But it is not sufficient.

Ex:

consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Let $a_n = \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is not a convergent series.

3.2 Series with non-negative terms:

Theorem 3.2.A:

If $\sum_{n=1}^{\infty} a_n$ is a series of nonnegative numbers with $S_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$, then

(a) $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded.

(b) $\sum_{n=1}^{\infty} a_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ is not bounded.

Proof:

Given $\sum_{n=1}^{\infty} a_n$ be a series of nonnegative numbers.

(i) $a_n \geq 0, \quad \forall n \in I.$

Let $S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$

$S_{n+1} = S_n + a_{n+1} \geq S_n$.

$\therefore S_n \leq S_{n+1} \quad \forall n \in I$.

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ is a nondecreasing sequence.

If the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded, $\{S_n\}_{n=1}^{\infty}$ is bounded above.

Hence by the theorem,

"A non decreasing sequence which is bounded above is convergent."

$\therefore$ We have $\{S_n\}_{n=1}^{\infty}$ is convergent.

Hence $\sum_{n=1}^{\infty} a_n$ converges.

(67) If $\{s_n\}_{n=1}^{\infty}$ is not bounded then $\sum_{n=1}^{\infty} a_n$ diverge.

3.2 B : Theorem :

(a) If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

(b) If $x \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Proof :

(a) If $0 < x < 1$, then $\{x^n\}_{n=1}^{\infty}$ converges to 0.

(i) $\lim_{n \to \infty} x^n = 0$

Hence $S_n = 1 + x + x^2 + \dots + x^{n-1}$

$$S_n = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{1}{1-x} - \frac{x^n}{1-x}\right]$$

$$= \lim_{n \to \infty} \frac{1}{1-x} - \lim_{n \to \infty} \frac{x^n}{1-x}$$

$$= \frac{1}{1-x} - 0$$

$$= \frac{1}{1-x}$$

Hence $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

(b) If $x \geq 1$, then $\lim_{n \to \infty} x^n \neq 0$.

(ii) $\sum_{n=0}^{\infty} x^n$ does not satisfy, the necessary condition for a series to converge. (Since the necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to converge is that $\lim_{n \to \infty} a_n = 0$)

Hence $\sum_{n=0}^{\infty} x^n$ diverges.

Theorem: 3.2.c:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

[$\sum_{n=1}^{\infty} \frac{1}{n}$ is known as harmonic Series].

Proof:

Let $S_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, $\forall n \in I$

Consider the subsequence $S_1, S_2, S_4, S_8, \ldots S_{2^n}$ of $\{S_n\}_{n=1}^{\infty}$.

Hence $S_1 = 1$

$S_2 = 1+\frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$

$S_{2^2} = S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

$= S_2 + \frac{1}{3}+\frac{1}{4}$

$> S_2 + \frac{1}{4}+\frac{1}{4}$

$> \frac{3}{2}+\frac{1}{2}$

$> \frac{4}{2}$

$S_4 > 2$ (or) $\frac{4}{2}$

$S_{2^3} = S_8 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

$= S_4 + \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

$> S_4 + \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$

$> 2+\frac{4}{8}$

$> 2+\frac{1}{2}$

$S_{2^3} > \frac{5}{2}$ (or) $\frac{3+2}{2}$

Hence assume that $S_{2^k} > \frac{k+2}{2}$

$$S_{2^{k+1}} = S_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}$$

$$> S_{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}$$

$$> \frac{k+2}{2} + \frac{2^k}{2^{k+1}}$$

$$> \frac{k+2}{2} + \frac{2^k}{2^k \cdot 2}$$

$$S_{2^{k+1}} = \frac{k+2}{2} + \frac{1}{2} = \frac{k+2+1}{2} = \frac{(k+1)+2}{2}$$

Hence by induction hypothesis,

$$S_{2^n} > \frac{n+2}{2} \quad \forall n \in I.$$

Hence the subsequence $\{S_{2^n}\}_{n=1}^{\infty}$ diverges to ∞.

(ii) A subsequence of $\{S_n\}_{n=1}^{\infty}$ diverges to ∞.

Hence by the theorem,

If a sequence converges then it subsequence also converges to the same limit.

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ diverges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Hence the proof:

3.2 D : Note :

(1) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, we sometimes write $\sum_{n=1}^{\infty} a_n < \infty$. —①

(b) If $\sum_{n=1}^{\infty} a_n$ is a divergent series of nonnegative numbers, we some times write

$$\sum_{n=1}^{\infty} a_n = \infty . \quad \text{—} (2)$$

Then

$$\sum_{n=0}^{\infty} (1/2)^n < \infty ,$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty .$$

3.2E: Theorem:

If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers which converges to zero but for which $\sum_{n=1}^{\infty} \varepsilon_n a_n$ still diverges.

Proof:

Let $S_n = a_1 + a_2 + \cdots + a_n$.

We first show that the series

$$\sum_{k=1}^{\infty} \frac{(S_{k+1} - S_k)}{S_{k+1}} \quad \text{diverges.}$$

For any $m \in I$ choose $n \in I$ such that $S_{n+1} \geq 2S_m$

($\because$ by hypothesis $\{S_k\}_{k=1}^{\infty}$ diverges to infinity).

Now $\{S_k\}_{k=1}^{\infty}$ is non decreasing.

Hence

$$\sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{k+1}} \geq \sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{n+1}}$$

$$\geq \frac{1}{S_{n+1}} \left[(S_{m+1} - S_m) + (S_{m+2} - S_{m+1}) + \cdots + (S_{n+1} - S_n) \right]$$

$$\geq \frac{S_{n+1} - S_m}{S_{n+1}} \geq \frac{S_{n+1} - \frac{1}{2} S_{n+1}}{S_{n+1}}$$

$$\geq \frac{\frac{1}{2} S_{n+1}}{S_{n+1}}$$

$$\geq \frac{1}{2}.$$

Thus for any $m \in I$ there exists $n \in I$ such that

$$\sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{k+1}} \geq \frac{1}{2}.$$

The partial sums of the series $\sum_{k=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}}$ thus do not form a Cauchy sequence. $\therefore$ its divergent.

Hence

$$\sum_{n=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}} = \infty$$

But $S_{k+1} - S_k = a_{k+1}$.

Then

$$\sum_{n=1}^{\infty} \frac{a_{k+1}}{S_{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{S_k} = \infty.$$

Let $\xi_k = \frac{1}{S_k}$.

Then $\xi_k \to 0$ as $k \to \infty$.

$\therefore \sum_{k=2}^{\infty} \xi_k a_k = \infty$. Hence $\sum_{n=1}^{\infty} \xi_n a_n$ still divergent.

3.3: Alternating Series:

Definition:

An alternating series is an infinite series whose terms alternate in sign.

Ex: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\cdots$

$1 - 2 + 3 - 4 + \cdots\cdots$

are all alternating series.

3.3 A Theorem: (Leibnitz's theorem)

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive number such that,

a) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ (that is $\{a_n\}_{n=1}^{\infty}$ is nonincreasing) and

b) $\lim\limits_{n \to \infty} a_n = 0$

Then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Proof:

Consider the first partial sums with odd index $S_1, S_3, S_5, \cdots\cdots$

We have $S_3 = S_1 - a_2 + a_3$

Since by (a) $a_3 \leq a_2 \leq a_1$

$\Rightarrow S_3 \leq S_1$

Then for any $n \in I$, we have

$S_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1}$

$S_{2n+1} \leq S_{2n-1}$ $[\because a_{2n} \geq a_{2n+1} \Rightarrow a_{2n} - a_{2n+1} \geq 0]$

$\therefore S_{2n-1} \geq S_{2n+1} \quad \forall n \in I.$

$\therefore \{S_{2n-1}\}_{n=1}^{\infty}$ is non increasing.

But $S_{2n-1} = (a_1 - a_2) + a_3 - \cdots + a_{2n-3} - a_{2n-2} + a_{2n-1}$

$= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2})$

$+ a_{2n-1}$

Since $a_1 \geq a_2$, $a_3 \geq a_4, \ldots$

$\therefore a_1 - a_2 \geq 0$

$a_3 - a_4 \geq 0, \ldots$

Also

$a_{2n-1} \geq 0$

$\therefore S_{2n-1} > 0 \quad \forall n \in I.$

Hence the non increasing subsequence $\{S_{2n-1}\}_{n=1}^{\infty}$ is bounded below.

Hence $\{S_{2n-1}\}_{n=1}^{\infty}$ is convergent.

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Now consider the partial sums $S_2, S_4, S_6, \ldots$

$S_2 = a_1 - a_2$

$S_4 = S_2 + a_3 - a_4$

$\geq S_2 + 0 \quad [a_3 - a_4 \geq 0]$

In general

$S_{2n+2} \geq S_{2n} + (a_{2n+1} - a_{2n+2})$

$S_{2n+2} \geq S_{2n} \quad [\because a_{2n+1} - a_{2n+2} \geq 0]$

$\Rightarrow S_{2n} \leq S_{2n+2}$

$\therefore$ The sequence $\{S_{2n}\}_{n=1}^{\infty}$ is non decreasing.

Also $S_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n-2} + a_{2n-1} - a_{2n}$

$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$

$S_{2n} \leq a_1$ $\left[\because a_2 - a_3 \geq 0 \right.$

$a_4 - a_5 \geq 0$

$\vdots$

$\left. a_{2n-2} - a_{2n-1} \geq 0, a_{2n} \geq 0 \right]$.

$\therefore$ The non decreasing sequence $\{S_{2n}\}_{n=1}^{\infty}$ is bounded above.

Hence $\{S_{2n}\}_{n=1}^{\infty}$ is convergent.

" Since a nondecreasing sequence which is bounded above is convergent."

Now let $M = \lim_{n \to \infty} S_{2n-1}$ and

let $L = \lim_{n \to \infty} S_{2n}$.

Then since

$$a_{2n} = S_{2n} - S_{2n-1}$$

Then by (b) we have

$$0 = \lim_{n \to \infty} a_{2n} = \lim_{n \to \infty} (S_{2n} - S_{2n-1})$$

$$= \lim_{n \to \infty} S_{2n} - \lim_{n \to \infty} S_{2n-1}$$

$$0 = L - M$$

$$\therefore L = M.$$

So both $\{S_{2n}\}_{n=1}^{\infty}$ and $\{S_{2n-1}\}_{n=1}^{\infty}$ converge to L.

Hence that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is converge to L.

Hence the theorem.

3.4: Conditional convergence and absolute convergence

The Series

$$\left.\begin{array}{l} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \\ \text{and } 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots \end{array}\right\} \text{both converge.}$$

3.4 A Definition:

Absolutely convergent:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Conditionally convergent:

Let $\sum_{n=1}^{\infty} a_n$ be a Series of real numbers.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally;

3.4 B: Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ Converges.

Proof:

Let $S_n = a_1 + a_2 + \cdots + a_n$.

We wish to prove that $\{S_n\}_{n=1}^{\infty}$ converges. since

By known theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers, then $\{s_n\}_{n=1}^{\infty}$ is convergent."

It is enough to show that $\{s_n\}_{n=1}^{\infty}$ is cauchy.

By hypothesis

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

and thus $\{t_n\}_{n=1}^{\infty}$ converges where $t_n = |a_1| + \dots + |a_n|$.

By known theorem,

"If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ converges, then $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence."

$\therefore \{t_n\}_{n=1}^{\infty}$ is cauchy.

Thus given $\varepsilon > 0$, there exists $N \in I$ ∋

$$|t_m - t_n| < \varepsilon \quad (m, n \geq N).$$

But (if $m > n$),

$$|s_m - s_n| = \cancel{|a_{n+1}}$$

$$= |(a_1 + a_2 + \dots + a_n + a_{n+1} + \dots + a_m) - (a_1 + a_2 + \dots + a_n)|$$

$$= |a_{n+1} + \dots + a_m|$$

$$\leq |a_{n+1}| + \dots + |a_m|$$

$$|s_m - s_n| \leq |t_m - t_n|$$

$$|s_m - s_n| < \varepsilon \quad (m, n \geq N).$$

This proves that $\{s_n\}_{n=1}^{\infty}$ is cauchy.

Hence the proof.

3.4 C:

Note:

1. If $\sum_{n=1}^{\infty} a_n$ is a series of real numbers, let

$$p_n = a_n \quad \text{if } a_n > 0$$

$$p_n = 0 \quad \text{if } a_n \leq 0$$

Ex:

for the series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$

$p_1 = 1$, $p_3 = \frac{1}{3}$, $p_{2n-1} = \frac{1}{2n-1}$ while

$p_2 = 0$, $p_4 = 1 \cdots = 0$.

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2. $$q_n = a_n \quad \text{if } a_n \leq 0$$

$$q_n = 0 \quad \text{if } a_n > 0$$

The p_n are thus the positive terms of $\sum_{n=1}^{\infty} a_n$ while the q_n are negative terms. Then

$$p_n = \max(a_n, 0), \quad q_n = \min(a_n, 0).$$

Since by known theorem

"
$$\max(a, b) = \frac{|a-b| + a + b}{2}$$

$$\min(a, b) = -\frac{|a-b| + a + b}{2} \text{"}$$

$$\therefore \max(a_n, 0) = p_n = \frac{|a_n - 0| + a_n + 0}{2}$$

$$= \frac{|a_n| + a_n}{2}$$

$$\boxed{\therefore 2p_n = |a_n| + a_n} \quad \text{—— (1)}$$

$\min(a_n, 0) = q_n = \dfrac{-|a_n - 0| + a_n + 0}{2}$

$= -|a| + a_n / 2$

$q_n = \dfrac{a_n - |a_n|}{2}$

$\boxed{2q_n = a_n - |a_n|}$ —— (2)

$\therefore$ From eq (1) & (2) we have

$2p_n + 2q_n = a_n + \cancel{|a_n|} + a_n - \cancel{|a_n|}$

$\cancel{2}(p_n + q_n) = \cancel{2}a_n$

$p_n + q_n = a_n$

Theorem:

(a) If $\sum_{n=1}^{\infty} a_n$ converges absolutely then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge. However

(b) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

Proof:

(a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ both converge. Then by known theorem

"If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B then, $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to A+B." —— (*)

So $\sum_{n=1}^{\infty} (a_n + |a_n|)$.

Thus From eq (1) & (2)

$\sum_{n=1}^{\infty} 2p_n$ converges.

By known Theorem ~~(again)~~ ⊛

This implies the convergence of $\sum_{n=1}^{\infty} p_n$.

The series $\sum_{n=1}^{\infty} q_n$ is convergent similarly.

(b) We assume that $\sum_{n=1}^{\infty} a_n$ converges but that $\sum_{n=1}^{\infty} |a_n|$ diverges.

From eq ① & ②

we have $|a_n| = 2p_n - a_n$

If $\sum_{n=1}^{\infty} p_n$ converged then by known theorem ⊛

$$\sum_{n=1}^{\infty} (2p_n - a_n) = \sum_{n=1}^{\infty} |a_n|,$$

Contraducting our assumption.

Hence $\sum_{n=1}^{\infty} p_n$ diverges.

Similary $\sum_{n=1}^{\infty} q_n$ diverges.

Hence the proof.

3.5: Rearrangement of Series:

A Rearrangement of a series $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} b_n$ whose terms are the same as those of $\sum_{n=1}^{\infty} a_n$ but occur in different order.

3.5C : Definition : Rearrangement :

Let $N = \{n_i\}_{i=1}^{\infty}$ be a sequence of positive integers where each positive integer occurs exactly once among the n_i.

If $\sum_{n=1}^{\infty} a_n$ is a series of real numbers and if

$\boxed{b_i = a_{n_i}}$ $(i \in I)$.

Then $\sum_{i=1}^{\infty} b_i$ is called a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Note:

(*) $\sum_{i=1}^{\infty} a_i$ is also a rearrangement of $\sum_{n=1}^{\infty} b_n$ if $\sum_{n=1}^{\infty} b_n$ is ~~a~~ rearrangement of $\sum_{n=1}^{\infty} a_n$.

3.5D : Theorem :

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series of real numbers. Then for any $x \in R$ there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ which converges to x.

3.5 E : Lemma :

If $\sum_{n=1}^{\infty} a_n$ is a series of nonnegative numbers which converges to $A \in R$ and $\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ converges, and $\sum_{n=1}^{\infty} b_n = A$.

Proof :

For each $N \in I$,

let $S_N = b_1 + b_2 + \cdots + b_N$.

Since $b_i = a_{n_i}$ for some sequence $\{n_i\}_{i=1}^{\infty}$
we have

$b_1 = a_{n_1}$,

$b_2 = a_{n_2}$,

$\vdots$

$b_N = a_{n_N}$

Let $M = \max(n_1, n_2, \ldots, n_N)$.

Then $S_N \leq a_1 + a_2 + \cdots + a_M \leq A$.

$S_N \leq A$.

Thus by known theorem

" $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded."

Then $\sum_{n=1}^{\infty} b_n$ converges to some $B \in \mathbb{R}$.

But $B = \lim_{n\to\infty} S_N$ and by known theorem

" If $S_n \leq t_n$ $(n \in \mathbb{J})$ and if $\lim_{n\to\infty} S_n = L$, $\lim_{n\to\infty} t_n = M$, then $L \leq M$."

$\therefore$ $\boxed{B \leq A}$.

(That is $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$).

But since $\sum_{n=1}^{\infty} a_n$ is also a rearrangement of $\sum_{n=1}^{\infty} b_n$,

$\therefore$ The same reasoning of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ reversed would be $\boxed{A \leq B}$.

Hence $A = B$.

$\therefore \sum_{n=1}^{\infty} b_n = A$.

3.5 F: Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges absolutely to A, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converges absolutely to A.

Proof:

We know that $a_n = p_n + q_n$.

By known theorem

"If $\sum_{n=1}^{\infty} a_n$ converges absolutely then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge."

$\therefore$ Both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converges.

$\therefore \sum_{n=1}^{\infty} p_n = P$ and $\sum_{n=1}^{\infty} q_n = Q$. $(Q \leq 0)$.

Then $A = P + Q$

For some $\{n_i\}_{i=1}^{\infty}$ we have

$b_i = a_{n_i} = p_{n_i} + q_{n_i}$.

Hence by known theorem

$\sum_{i=1}^{\infty} p_{n_i}$ converges and

$\sum_{i=1}^{\infty} p_{n_i} = P$.

Similarly,

$$\sum_{i=1}^{\infty} q_{n_i} = Q.$$

since $b_i = p_{n_i} + q_{n_i}$

By known theorem

"If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converge to B, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $A+B$."

$\therefore \sum_{i=1}^{\infty} b_i$ converges and

$$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} p_{n_i} + \sum_{i=1}^{\infty} q_{n_i}$$

$$= P + Q$$

$$= A.$$

since $b_i = p_{n_i} + q_{n_i}$

we have $|b_i| = |p_{n_i} + q_{n_i}|$

$$\leq |p_{n_i}| + |q_{n_i}|$$

$$\leq p_{n_i} - q_{n_i}$$

Thus for any $N \in I_+$,

$$|b_1| + |b_2| + \cdots + |b_N| \leq \sum_{i=1}^{N} p_{n_i} - \sum_{i=1}^{N} q_{n_i}$$

$$\leq \sum_{i=1}^{\infty} p_{n_i} - \sum_{i=1}^{\infty} q_{n_i}$$

$$\leq P - Q$$

$\therefore$ The partial sums of $\sum_{i=1}^{\infty} |b_i|$ are all bounded above by $P - Q$.

$\therefore$ Hence $\sum_{i=1}^{\infty} |b_i| < \infty$.

$\therefore$ The rearrangement $\sum_{n=1}^{\infty} b_n$ is also converges absolutely to A.

Hence the proof.

Theorem 3.5.G :

If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely to A and B respectively, then $AB = C$ where $C = \sum_{n=0}^{\infty} c_n$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(k=0,1,2,\ldots)$

Proof:

Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely to A and B.

Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $k=0,1,\ldots$

To prove $C = AB$.

For $k = 0, 1, 2, \ldots$ we have

$$|c_k| \leq |a_0 b_k| + |a_1 b_{k-1}| + \cdots + |a_k b_0|.$$

Thus for any n,

$$|c_0| + |c_1| + \cdots + |c_n|$$

$$\leq |a_0 b_0| + (|a_0 b_1| + |a_1 b_0|) + \cdots + (|a_0 b_n| + |a_1 b_{n-1}| + \cdots + |a_n b_0|)$$

$$\leq (|a_0| + \cdots + |a_n|)(|b_0| + \cdots + |b_n|)$$

$$< \left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{k=0}^{\infty} |b_k|\right).$$

The sequence of partial sums of $\sum_{k=0}^{\infty} |c_k|$ is thus bounded above and hence

$$\sum_{k=0}^{\infty} |c_k| < \infty.$$

The forgoing inequalities also show the absolute convergence of the series

$$a_0b_0 + a_0b_1 + a_1b_0 + a_0b_2 + a_1b_1 + a_2b_0 + a_0b_3 + \cdots \quad (1)$$

whose sum is $\sum_{k=0}^{\infty} c_k$.

By known theorem we may rearrange the terms in (1) to obtain

$$\sum_{k=0}^{\infty} c_k = [a_0b_0] + [a_0b_1 + a_1b_0 + a_1b_1] + [a_0b_2 + a_2b_0 + a_1b_2 + a_2b_1 + a_2b_2] + \cdots \quad (2)$$

Inside the nth bracket $(n=0,1,2,\ldots)$ on the right of (2) are all products a_jb_k where either j or k is equal to n and neither j nor k is greater than n.

Let us examine the sum of the terms in each bracket.

If $A_n = a_0 + a_1 + \cdots + a_n$ and

$B_n = b_0 + b_1 + \cdots + b_n$

We have $a_0b_0 = A_0B_0$.

$$a_0b_1 + a_1b_0 + a_1b_1 = (a_0+a_1)(b_0+b_1) - a_0b_0$$
$$= A_1B_1 - A_0B_0$$

$$a_0 b_2 + a_2 b_0 + a_1 b_2 + a_2 b_1 + a_2 b_2 = (a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - (a_0 + a_1)(b_0 + b_1)$$

$$= A_2 B_2 - A_1 B_1$$

and in general, for $n \geq 1$, the quantity in n^{th} braket of ② is equal to $A_n B_n - A_{n-1} B_{n-1}$.

$\therefore$ ② becomes:

$$\sum_{k=0}^{\infty} C_k = [A_0 B_0] + [A_1 B_1 - A_0 B_0] + \cdots + [A_n B_n - A_{n-1} B_{n-1}]$$

$$= A_n B_n$$

$$\boxed{C = AB} \text{ as } n \to \infty.$$

3.54: corollary:

If for some $x \in R$ the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are absolutely convergent, then

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proof:

Let $A_n = a_n x^n$, $B_n = b_n x^n$ Then By known thm

$$\left(\sum_{n=0}^{\infty} A_n\right)\left(\sum_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty} C_n \quad \text{——— ①}$$

where $C_n = \sum_{k=0}^{n} A_k B_{n-k}$

$$= \sum_{k=0}^{n} a_k x^k b_{n-k} x^{n-k}$$

$= x^n \sum_{k=0}^{n} a_k b_{n-k}$

$= C_n x^n$

Hence the proof.

3.6 Test for absolute convergence

3.6 A : Definition: Test for Absolute convergence:

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers. We shall say that $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ if there exists $N \in I$ such that

$$|a_n| \leq |b_n| \qquad (n \geq N).$$

It can be written as $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$.

3.6 B : Theorem [Comparison Test]

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ where $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ also converges absolutely. Symbolically, if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |b_n| < \infty$, then $\sum_{n=1}^{\infty} |a_n| < \infty$.

Proof:

Let $M = \sum_{n=1}^{\infty} |b_n|$

Since $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$.

Then there exists $N \in I$ &

$|a_n| \leq |b_n| \ \forall \ n \geq N.$

Let $S_n = |a_1| + |a_2| + \cdots + |a_n|$, then for $n \geq N$,

$$S_n \leq |a_1| + |a_2| + \cdots + |a_{N-1}| + |a_N| + |a_{N+1}| + \cdots + |a_n|$$
$$\leq |a_1| + |a_2| + \cdots + |a_{N-1}| + |b_N| + |b_{N+1}| + \cdots + |b_n|$$

$$S_n \leq |a_1| + \cdots + |a_{N-1}| + M.$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of $\sum_{n=1}^{\infty} |a_n|$ is bounded above.

Hence by the theorem,

" Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers and $S_n = a_1 + a_2 + \cdots + a_n \ \forall \ n \in I$.

if $\{S_n\}_{n=1}^{\infty}$ is bounded then $\sum_{n=1}^{\infty} a_n$ converges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Hence $\sum_{n=1}^{\infty} a_n$ also converges absolutely.

Theorem 3.6 D:

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n| = \infty$, then $\sum_{n=1}^{\infty} |b_n| = \infty$. (That is, if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n| = \infty$, then $\sum_{n=1}^{\infty} |b_n| = \infty$)

Proof:

Since $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$, there exists $N_1 \in I$ $\ni$

$|a_n| \leq |b_n|, \forall n \geq N_1$

Let $t_n = |b_1| + |b_2| + \cdots + |b_n|$

Then for $n \geq N_1$

$$t_n = |b_1| + \cdots + |b_{N_1-1}| + |b_{N_1}| + |b_{N_1+1}| + \cdots + |b_n|$$
$$\geq |b_1| + \cdots + |b_{N-1}| + |a_{N_1}| + |a_{N_1+1}| + \cdots + |a_n| \quad \text{--- (1)}$$

since $\sum_{n=1}^{\infty} |a_n| = \infty$.

if $S_n = |a_1| + |a_2| + \cdots + |a_n|$

Hence for a real number $M > 0$, there exists $N_2 \in \mathbb{I}$ such that

$$S_n > M + [|a_1| + |a_2| + \cdots + |a_{N}|], \forall n \geq N_2 \quad \text{--- (2)}$$

Let $N = \max(N_1, N_2)$, then for $n \geq N$,

from (2)

$$S_n > M + [|a_1| + |a_2| + \cdots + |a_{N_1}|]$$

$|a_1| + |a_2| + \cdots + |a_{N_1}| + |a_{N_1+1}| + \cdots + |a_n| \geq M + [|a_1| + \cdots + |a_{N_1}|]$

$|a_{N+1}| + |a_{N+2}| + \cdots + |a_n| > M$ --- (3)

using eq (3) in (1) we have,

$$t_n > |b_1| + \cdots + |b_{N-1}| + M, \quad \forall n \geq N.$$

Hence $\sum_{n=1}^{\infty} |b_n| = \infty$.

3.6 E Theorem:

(a) If $\sum_{n=1}^{\infty} b_n$ converges absolutely and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $\sum_{n=1}^{\infty} |a_n| = \infty$ and if $\lim_{n\to\infty} |a_n|/|b_n|$ exists, then $\sum_{n=1}^{\infty} |b_n| = \infty$.

Proof:

Since $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists.

$\therefore \left\{ \frac{|a_n|}{|b_n|} \right\}_{n=1}^{\infty}$ converges, Hence $\left\{ \frac{|a_n|}{|b_n|} \right\}_{n=1}^{\infty}$ is bounded.

[since a convergent sequence is bounded].

$\Rightarrow$ There exists $M > 0$, such that $\frac{|a_n|}{|b_n|} \leq M \ \forall n \in I$

$\Rightarrow |a_n| \leq M \cdot |b_n| \ \forall n \in I$ — (1)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is dominated by the series $\sum_{n=1}^{\infty} M \cdot b_n$.

By theorem "comparison test"

$\sum_{n=1}^{\infty} |a_n| < \infty$.

(b) As in the proof of (a) we have

$|a_n| \leq M |b_n|$.

$\frac{1}{M} |a_n| \leq |b_n|$

so that $\sum_{n=1}^{\infty} |b_n|$ dominates $\sum_{n=1}^{\infty} (1/M) \cdot |a_n|$.

By theorem,

if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n| = \infty$ then

$\sum_{n=1}^{\infty} |b_n| = \infty$.

$\therefore$ we have $\sum_{n=1}^{\infty} |b_n| = \infty$.

Hence the proof.

Theorem: 3.6.F [Ratio Test]

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero real numbers and let

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \quad A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

(so that $a \leq A$). Then

(a) If $A < 1$, then $\sum_{n=1}^{\infty} |a_n| < \infty$,

(b) If $a > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

(c) If $a \leq 1 \leq A$, then the test fails.

Proof:

(a) If $A < 1$,

choose any B such that $A < B < 1$.

Then $B = A + \varepsilon$, for some $\varepsilon > 0$ so by Known theorem

"If $\limsup_{n\to\infty} s_n = M$, then for any $\varepsilon > 0$, then $s_n < M + \varepsilon$ for all but a finite number of values of n."

Then there exists $N \in I$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \le B \quad (n \ge N).$$

Then $\left|\frac{a_{N+1}}{a_N}\right| \le B,$

$\left|\frac{a_{N+2}}{a_{N+1}}\right| \le B,$ and so

$$\left|\frac{a_{N+2}}{a_N}\right| = \left|\frac{a_{N+2}}{a_{N+1}}\right| \cdot \left|\frac{a_{N+1}}{a_N}\right|$$

$$\le B^2$$

for any $k \ge 0$, we have similarly

$$\left|\frac{a_{N+k}}{a_N}\right| = \left|\frac{a_{N+k}}{a_{N+k-1}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right|$$

$$\frac{|a_{N+k}|}{|a_N|} \le B^k.$$

Thus $|a_{N+k}| \le |a_N| B^k \quad (k = 0, 1, 2, \ldots)$

But $\sum_{k=0}^{\infty} |a_N| . B^k$ converges since $0 < B < 1$.

Hence $\sum_{k=0}^{\infty} |a_N| B^k$ converges

$\therefore \sum_{k=0}^{\infty} |a_{N+k}|$ converges.

(ii) $|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots$ converges

$\therefore |a_1| + |a_2| + \cdots + |a_{N-1}| + |a_N| \cdots$ converges

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Hence if $A < 1$, then $\sum_{n=1}^{\infty} |a_n| < \infty$.

b) If $a > 1$, choose b such that $a > b > 1$.

Let $\varepsilon = a - b > 0$.

$\Rightarrow b = a - \varepsilon$

By the theorem,

"If $M = \lim_{n \to \infty} \inf S_n$, then $S_n > M - \varepsilon$, $\forall$ but a finite number of values of n".

$\Rightarrow$ There exists an $N \in I$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > a - \varepsilon$$

$$= b > 1 \quad \forall (n \geq N)$$

$\Rightarrow |a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$

Hence $\{a_n\}_{n=1}^{\infty}$ cannot converge to 0.

i.e.) $\lim_{n \to \infty} a_n \neq 0$.

Hence $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If $a \leq 1 \leq A$, then the test fails.

To prove this, consider the example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \quad \therefore \sum_{n=1}^{\infty} a_{n+1} = \frac{1}{n+1}$$

$$\therefore \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \frac{n}{n+1} = \frac{n}{n(1+1/n)}$$

$$= \frac{1}{1+1/n}$$

$$\therefore \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

$\therefore A = a = 1$: The series diverges.

Hence the proof.

Theorem : 3.6.4 : Root Test

If $\limsup_{n \to \infty} \sqrt[n]{a_n} = A$ then the series of real numbers $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely if $A < 1$

(b) diverges if $A > 1$

(c) If $A = 1$ the test fails.

Proof:

a) If $A < 1$, choose B so that $A < B < 1$.

Let $\varepsilon = B - A > 0$.

$\Rightarrow B = A + \varepsilon$

Since $A = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$

By the theorem,

If $M = \limsup_{n \to \infty} s_n$, then $s_n < M + \varepsilon$, $\forall$ but a finite number of values of n and $s_n > M - \varepsilon$ $\forall$ infinitely many values of n,

Then there exists $N \in I$ such that

$\sqrt[n]{|a_n|} < A + \varepsilon \quad \forall n \geq N.$

$\Rightarrow |a_n| < B^n, \quad \forall n \geq N.$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| \ll \sum_{n=1}^{\infty} B^n$ and

$\sum_{n=1}^{\infty} B^n$ is absolutely convergent $[\because 0 < B < 1]$

By the theorem,

" Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers.

If $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and if $\sum_{n=1}^{\infty} |b_n| < \infty$, then

$\sum_{n=1}^{\infty} |a_n| < \infty$. "

$\therefore \sum_{n=1}^{\infty} a_n$ converges absolutely.

b) If $A > 1$, choose B so that $A > B > 1$.

Let $\varepsilon = A - B > 0$, then

$B = A - \varepsilon$.

Again by the theorem

$\sqrt[n]{|a_n|} > A - \varepsilon$ for infinitely many values of n.

$\Rightarrow \sqrt[n]{|a_n|} > B$ "

$\sqrt[n]{|a_n|} > 1$ "

$\Rightarrow \sqrt{|a_n|} > 1$ "

$\therefore \{a_n\}_{n=1}^{\infty}$ does not converge to zero.

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ does not satisfy the necessary condition for convergent of a series.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

c) To prove $A = 1$, the test fails.

Consider the example, $a_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\sqrt[n]{|a_n|} = \left(\frac{1}{n}\right)^{1/n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1/n\right)^{1/n}$$

$$= \lim_{n\to\infty} e^{\log (1/n)^{1/n}}$$

$$= \lim_{n\to\infty} e^{1/n \log 1/n}$$

$$= \lim_{n\to\infty} e^{1/n [\log 1 - \log n]}$$

$$= \lim_{n\to\infty} e^{1/n (0 - \log n)}$$

$$= \lim_{n\to\infty} e^{-\frac{\log n}{n}}$$

$$= e^0$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

$\therefore \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

(ie) $A = 1$

Hence the test fails.

3.7. Series whose terms form a non increasing sequence:

3.7A Theorem: Cauchy condensation test:

If $\{a_n\}_{n=1}^{\infty}$ is a non increasing sequence of positive numbers and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof:

We have $a_1 \le a_1$

$$a_2+a_3 \le a_2+a_2 = 2a_2$$

$$a_4+a_5+a_6+a_7 \le a_4+a_4+a_4+a_4 = 4a_4 = 2^2 a_{2^2}.$$

and for any $n \in I$,

$$a_{2^n} + a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^{n+1}-1} \le 2^n a_{2^n}.$$

From these inequalities it follows that

$$\sum_{k=1}^{2^{n+1}-1} a_k \le \sum_{k=0}^{n} 2^k a_{2^k} \le \sum_{k=0}^{\infty} 2^k a_{2^k}.$$

Hence for any $m \in I$, we have

$$\sum_{k=1}^{m} a_k \le \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \text{— (1)}$$

By hypothesis,

since $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges,

Then eq (1) $\Rightarrow$ The sequence of n^{th} partial sums of the series $\sum_{n=1}^{\infty} a_n$ is ~~not~~ bounded.

Hence $\sum_{n=1}^{\infty} a_n$ converges.

By the theorem

"Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers,

$S_n = a_1 + a_2 + \cdots + a_n$

if $\{S_n\}_{n=1}^{\infty}$ is bounded then $\sum_{n=1}^{\infty} a_n$ converges".

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Hence the proof.

3.7 B Theorem:

If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof:

We have $a_3 + a_4 \geq a_4 + a_4 = 2a_4$

$$= \frac{1}{2}\left[2^2 a_{2^2}\right]$$

$a_5 + a_6 + a_7 + a_8 \geq a_8 + a_8 + a_8 + a_8$

$$= 4a_8$$

$$= \frac{1}{2}\left[2^3 a_{2^3}\right]$$

In general

$$a_{2^n+1} + \cdots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}}$$
$$= \frac{1}{2}\left[2^{n+1} a_{2^{n+1}}\right].$$

From these inequalities it follows a_k.

$$\sum_{k=3}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=1}^{n+1} 2^{k+1} a_{2^{k+1}}$$
$$\geq \frac{1}{2} \sum_{k=2}^{n+1} 2^k a_{2^k} \quad \text{---} (1)$$

Since $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges.

Eq (1) implies the sequence of n^{th} partial sums of the series $\sum_{n=1}^{\infty} a_n$ is not bounded.

Hence $\sum_{n=1}^{\infty} a_n$ diverges.

By the theorem,

"Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers, if $\{S_n\}_{n=1}^{\infty}$ is not bounded then $\sum_{n=1}^{\infty} a_n$ diverges."

$\therefore \sum_{n=1}^{\infty} a_n$ diverges.

Hence the proof.

Corollary 3.7.C

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Proof:

Let $a_n = \frac{1}{n^2}$

$$\therefore \sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right)^2$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

By the theorem

"If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$",

$\therefore$ we have $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges to $\frac{1}{1-\frac{1}{2}} = \frac{2}{1} = 2$

Hence by cauchy condensation test, we have

$\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges.

Theorem: 3.7.D Abel's theorem [or]

Pringsheim's theorem.

If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

Proof:

Let $S_n = a_1 + a_2 + \dots + a_n$

If $\sum_{n=1}^{\infty} a_n = A$.

Then $\lim_{n\to\infty} S_n = A$

$\therefore \lim_{n\to\infty} S_{2n} = A$.

Thus $\lim_{n\to\infty} (S_{2n} - S_n) = A - A = 0$.

Now $S_{2n} - S_n = \cancel{a_1 + a_2 + \cdots + a_n} + a_{n+1} + a_{n+2} + \cdots + a_{2n}$
$- \cancel{(a_1 + a_2 + \cdots + a_n)}$

$= a_{n+1} + a_{n+2} + \cdots + a_{2n}$

$\geq a_{2n} + a_{2n} + a_{2n} + \cdots + a_{2n}$

$\geq n\, a_{2n}$.

and so $0 \leq n a_{2n} \leq S_{2n} - S_n$.

Thus $\lim_{n\to\infty} n a_{2n} = 0$.

$\therefore \lim_{n\to\infty} 2n a_{2n} = 0$. —— (1)

But $a_{2n+1} \leq a_{2n}$. Thus

$$(2n+1)\, a_{2n+1} \leq \frac{(2n+1)}{2n} 2n a_{2n}$$

$\lim_{n\to\infty} (2n+1) a_{2n+1} = 0$ —— (2)

$\therefore$ From (1) & (2) we have

$\lim_{n\to\infty} n a_n = 0$.

Hence the proof.

3.10. The class ℓ^2

Definition 3.10 A:

The class ℓ^2 is the class of all sequences $S = \{S_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} S_n^2 < \infty$.

3.10 B: Theorem: The Schwarz Inequality.

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in ℓ^2, then $\sum_{n=1}^{\infty} s_n t_n$ is absolutely convergent and

$$\left| \sum_{n=1}^{\infty} s_n t_n \right| \le \left(\sum_{n=1}^{\infty} s_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} t_n^2 \right)^{1/2} \quad \text{—— (1)}$$

Proof:

We may assume that there is at least one s_n say $s_N \neq 0$.

otherwise the theorem is trivial.

For fixed $n \ge N$ and any $x \in \mathbb{R}$, we have

$$\sum_{k=1}^{n} (x s_k + t_k)^2 \ge 0$$

Expanding the parenthesis on the left we have

$$x^2 \sum_{k=1}^{n} s_k^2 + 2x \sum_{k=1}^{n} s_k t_k + \sum_{k=1}^{n} t_k^2 \ge 0.$$

This can be written

$$Ax^2 + Bx + C \ge 0.$$

where $A = \sum_{k=1}^{n} s_k^2 > 0$, $B = 2 \sum_{k=1}^{n} s_k t_k$,

$$C = \sum_{k=1}^{n} t_k^2 .$$

we know that the minimum value of

$Ax^2 + Bx + C$ $(A > 0)$ occurs when $x = -B/2A$.

If we set $x = -B/2A$,

we have $Ax^2 + Bx + C \geq 0$.

$$A\left(-B/2A\right)^2 + B\left(-B/2A\right) + C \geq 0 .$$

$$\cancel{A}\left(\frac{B^2}{4A^{\cancel{2}}}\right) + \frac{-B^2}{2A} + C \geq 0 .$$

$$\frac{B^2}{4A} - \frac{B^2}{2A} + C \geq 0 .$$

$$\frac{B^2 - 2B^2}{4A} + C \geq 0 .$$

$$-\frac{B^2}{4A} + C \geq 0 .$$

$$-B^2 + 4AC \geq 0 .$$

$$4AC \geq B^2$$

$$\therefore B^2 \leq 4AC .$$

But

$$\cancel{4}\left(\sum_{k=1}^{n} s_k t_k\right)^2 \leq \cancel{4}\left(\sum_{k=1}^{n} s_k^2\right) \cdot \left(\sum_{k=1}^{n} t_k^2\right) \quad (2)$$

If we replace s_k, t_k by $|s_k|, |t_k|$ in eq (2),

we obtain

$$\sum_{k=1}^{n} |s_k t_k| \le \left(\sum_{k=1}^{n} s_k^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

$$\le \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} \cdot \left(\sum_{k=1}^{\infty} t_k^2\right)^{1/2}$$

The sequence of partial sums of $\sum_{k=1}^{\infty} |s_k t_k|$ is thus bounded, and so $\sum_{k=1}^{\infty} |s_k t_k| < \infty$.

$\therefore \sum_{k=1}^{\infty} s_k t_k$ converges by known theorem.

If we now let n approach infinity in (2) we get eq (1).

$\therefore$ Hence

$$\left|\sum_{n=1}^{\infty} s_n t_n\right| \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}.$$

3.10 C. **Theorem:** (The Minkowski inequality).

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $s+t = \{s_n+t_n\}_{n=1}^{\infty}$ is in l^2 and

$$\left[\sum_{n=1}^{\infty} (s_n+t_n)^2\right]^{1/2} \le \left[\sum_{n=1}^{\infty} s_n^2\right]^{1/2} + \left[\sum_{n=1}^{\infty} t_n^2\right]^{1/2}$$

Proof:

By hypothesis, the series $\sum_{n=1}^{\infty} s_n^2$ and t_n^2 converge.

The series $\sum_{n=1}^{\infty} s_n t_n$ converge by known theorem. (Schwarz inequality).

Since $(s_n+t_n)^2 = s_n^2 + 2s_nt_n + t_n^2$.

By known theorem

$\sum_{n=1}^{\infty}(s_n+t_n)^2$ converges and

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 = \sum_{n=1}^{\infty}s_n^2 + 2\sum_{n=1}^{\infty}s_nt_n + \sum_{n=1}^{\infty}t_n^2$$

Applying the Schwarz inequality to the second sum on the right, we obtain

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \sum_{n=1}^{\infty}s_n^2 + 2\left(\sum_{n=1}^{\infty}s_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}t_n^2\right)^{1/2} + \sum_{n=1}^{\infty}t_n^2$$

and

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \left[\left(\sum_{n=1}^{\infty}s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty}t_n^2\right)^{1/2}\right]^2$$

Taking the square root on both sides we get

$$\left[\sum_{n=1}^{\infty}(s_n+t_n)^2\right]^{1/2} \le \left[\sum_{n=1}^{\infty}s_n^2\right]^{1/2} + \left[\sum_{n=1}^{\infty}t_n^2\right]^{1/2}.$$

3.10D. Definition: Norm:

If $s = \{s_n\}_{n=1}^{\infty}$ is an element of l^2 we define $\|s\|_2$ called the norm of s, as $\|s\|_2 = \left(\sum_{n=1}^{\infty}s_n^2\right)^{1/2}$.

3-10E: Theorem:

The norm for sequences in ℓ^2 has the following properties.

1. $\|s\|_2 \geq 0 \quad (s \in \ell^2)$

2. $\|s\|_2 = 0$ if and only if $s = \{0\}_{n=1}^{\infty}$.

3. $\|cs\|_2 = |c| \cdot \|s\|_2 \quad (c \in R, s \in \ell^2)$.

4. $\|s+t\|_2 \leq \|s\|_2 + \|t\|_2 \quad (s, t \in \ell^2)$.