Unit - IV

2.31. Relations :-

Definition :-

Any set of ordered pairs defines a binary relation.

* ~~The relation is den denoted by~~,

* Ex (i) : The relation "greater than" for real numbers, denoted by ">"

$$> = \{\langle x, y\rangle \,/\, x, y \text{ are real numbers and } x > y\}$$

Ex (ii) : The relation of father to his child can be described by a set, say F, of ordered pairs in which the first member is the name of the father and the second ~~member~~ the name of his child.

i.e, $F = \{\langle x, y\rangle \,/\, x \text{ is the father of } y\}$

(iii) $S = \{ \langle 2,4 \rangle, \langle 1,3 \rangle, \langle \lambda, 6 \rangle, \langle \text{Joan}, \mu \rangle \}$

(iv) $\mathbb{R}$ is the set of real numbers,

$Q = \{ \langle x^2, x \rangle \mid x \in \mathbb{R} \}$

Definition :- Domain of S

Let S be a binary relation. The set D(S) of all objects x such that for some y, $\langle x,y \rangle \in S$ is called the domain of S,

i.e, $D(S) = \{ x \mid (\exists y)(\langle x,y \rangle \in S) \}$

Range of S

Let S be a binary relation. The set R(S) of all objects y such that for some x, $\langle x,y \rangle \in S$ is called the range of S,

i.e, $R(S) = \{ y \mid (\exists x)(\langle x,y \rangle \in S) \}$

Ex:- $S = \{\langle 2,4\rangle, \langle 1,3\rangle, \langle \lambda,6\rangle, \langle \text{Jaan}, \mu\rangle\}$

$D(S) = \{2, 1, \lambda, \text{Jaan}\}$

$R(S) = \{4, 3, 6, \mu\}$.

Operations on Relations

* If R and S denote two relations, then $R \cup S$ defines a relation such that,

$$x(R \cup S)y \Leftrightarrow xRy \vee xSy$$

$R \cap S$ is a relation such that

$$x(R \cap S)y \Leftrightarrow xRy \wedge xSy$$

$R - S$ is also relation such that

$$x(R - S) \Leftrightarrow xRy \wedge x \not{S} y$$

and

$$x(\sim R)y \Leftrightarrow x \not{R} y$$

Example 1

(a) Let $X = \{1, 2, 3, 4\}$. If

$R = \{\langle x,y\rangle / x \in X \wedge y \in X \wedge ((x-y)$ is an Integral non-zero multiple of $2)\}$

$= \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

$S = \{\langle x,y\rangle / x \in X \wedge y \in X \wedge ((x-y)$ is an Integral non-zero multiple of $3)\}$

$= \{\langle 1,4\rangle, \langle 4,1\rangle\}$

$R \cup S = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 1,4\rangle, \langle 4,1\rangle\}$

$R \cap S = \phi$

(b) If $X = \{1, 2, 3, \dots\}$ what is $R \cap S$ as defined in (a) ?

Important Notes

$R = \{\langle 1,3\rangle, \langle 1,5\rangle, \langle 1,7\rangle, \langle 1,9\rangle, \dots$

$\langle 2,4\rangle, \langle 2,6\rangle, \langle 2,8\rangle, \langle 2,10\rangle, \dots$

$\langle 3,1\rangle, \langle 3,5\rangle, \langle 3,7\rangle, \langle 3,9\rangle, \dots$

$\langle 4,2\rangle, \langle 4,6\rangle, \langle 4,8\rangle, \langle 4,10\rangle \dots \}$

$S = \{\langle 1,4\rangle, \langle 1,7\rangle, \langle 1,10\rangle, \ldots$

$\langle 2,5\rangle, \langle 2,8\rangle, \langle 2,11\rangle, \ldots$

$\langle 3,6\rangle, \langle 3,9\rangle, \langle 3,12\rangle, \ldots$

$\langle 4,1\rangle, \langle 4,7\rangle, \langle 4,10\rangle \ldots \}$

$R \cap S = \{\langle 1,7\rangle, \ldots$

$\langle 2,8\rangle, \ldots$

$\langle 3,9\rangle \ldots$

$\langle 4,10\rangle \ldots \}$

$\therefore R \cap S = \{\langle x,y\rangle \mid x \in X \wedge y \in X \wedge ((x-y)$ is a nonzero multiple of $6)\}$

Problems :- (Home work - Try this)

1. Let $P = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,3\rangle\}$ and

$Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

Find $P \cup Q$, $P \cap Q$, $D(Q)$, $D(Q)$, $D(P \cup Q)$, $R(P)$, $R(Q)$, and $R(P \cap Q)$. Show that

$D(P \cup Q) = D(P) \cup D(Q)$

$R(P \cap Q) \subseteq R(P) \cap R(Q)$.

2.3.2. Properties of Binary Relations in a set.

Definition :- Reflexive

A binary relation in R in a set X is "reflexive" if, for every $x \in X$ xRx i.e, $\langle x, x \rangle \in R$.

(Or)

R is reflexive in X $\Leftrightarrow (x)(x \in X \rightarrow xRx)$

Ex:- * The relation $\leq$, $\geq$, $=$ are all reflexive in the set of real numbers.

* The relation $<$ and $>$ are not reflexive in the set of real numbers.

Symmetric

A relation R in a set X is "Symmetric" if, for every x and y in X, whenever xRy, then yRx.

ie,
R is symmetric in X

$\Leftrightarrow (x)(y)(x \in X \wedge y \in X \wedge xRy \rightarrow yRx)$

Transitive

A relation R in a set X is "transitive" if, for every x, y and z in X whenever xRy and yRz then xRz.

ie, R is transitive in X $\Leftrightarrow (x)(y)(z)(x \in X \wedge y \in X \wedge z \in X \wedge xRy \wedge yRz \rightarrow xRz)$

Irreflexive

A relation R in a set X is irreflexive, if, for every $x \in X$, $(x, x) \notin R$.

Anti-Symmetric

A relation R in a set X is antisymmetric if for every x and y in X whenever xRy and yRx then x=y.
ie, R is antisymmetric in X iff

$(x)(y)(x \in X \wedge y \in X \wedge xRy \wedge yRx \rightarrow x = y)$

2.3.8. Partial ordering.

Definition <u>Partial Ordering</u> A binary relation R in a set P is called a "partial order relation" or a "partial ordering" in P iff R is reflexive, antisymmetric and transitive.

* Denote a partial ordering by the symbol "$\leq$".
* If "$\leq$" is a partial ordering on P, then the ordered pair $(P, \leq)$ is called a "partially order set" or a "<u>Poset</u>"

Definition :-

Let $(P, \leq)$ be a partially ordered set. If for every $x, y \in P$ we have either $x \leq y \vee y \leq x$, then x is called a simple ordering or linear ordering on P. and $(P, \leq)$ is called a ~~tot~~ totally ordered (or) simply ordered set or a chain.

Note :-

* If R is partial ordering on P, then the converse of R, namely $\tilde{R}$ is also a partial ordering on P. If R is denoted by $\leq$ then $\tilde{R}$ is denoted by $\geq$.

i.e, if $(P, \leq)$ is a partially ordered set then $(P, \geq)$ is also a partially ordered set.

* $(P, \geq)$ is called the dual of $(P, \leq)$

* The relation "$<$" on P is denoted by,

for every $x, y \in P$ as

$$x < y \Leftrightarrow x \leq y \wedge x \neq y.$$

Similarly, the relation "$>$" on P is denoted by

for every $x, y \in P$ as

$$x > y \Leftrightarrow x \geq y \wedge x \neq y$$

Important Notes

Examples :-

1. Inclusion :- Let $P(A) = 2^A = X$ be the power set of A.

i.e., X is the set of subsets of A.

The relation "inclusion" ($\subseteq$) on X is a Partial ordering.

Let $A = \{a, b, c\}$

$X = P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, A\}$

$(X, \subseteq)$ is a partially ordered set.

2. Let $X = \{2, 3, 6, 8\}$ and let '$\leq$' be the relation "divides" on X.

Then $\leq = \{\langle 2,2\rangle, \langle 2,6\rangle, \langle 2,8\rangle, \langle 3,3\rangle, \langle 3,6\rangle, \langle 6,6\rangle, \langle 8,8\rangle\}$

and the relation "integral multiple of" is written as "$\geq$" is given by,

$\geq = \{\langle 2,2\rangle, \langle 3,3\rangle, \langle 6,6\rangle, \langle 8,8\rangle, \langle 8,2\rangle, \langle 6,2\rangle, \langle 6,3\rangle\}$

2.3.9. Partially Ordered set: Representation and Associated Terminology.

Covers:-

Let $(P, \leq)$ be a partially ordered set, an element $y \in P$ is said to "cover" an element $x \in P$ if $x < y$ and if there does not exists any element $z \in P$ such that $x \leq z$ and $z \leq y$

i.e,

$$y \text{ covers } x \Leftrightarrow (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$$

* **Hasse Diagram:-**

A partial ordering '$\leq$' on P can be represented by means of a diagram known as a "Hasse diagram" or "Partially ordered set diagram of $(P, \leq)$.

Note:- (1) Each element is represented by a small circle or a dot.

Important Notes

(i) The circle for $x \in P$ is drawn below the circle $y \in P$ if $x < y$, and a line is drawn between x and y if y covers x.

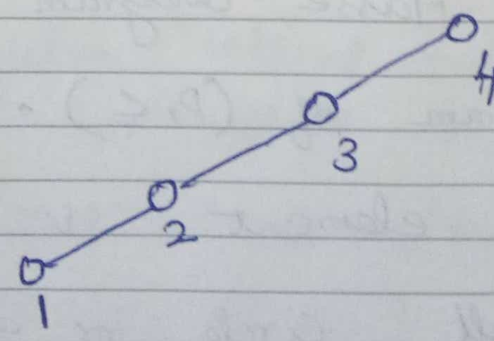
(ii) If $x < y$ but y does not cover x, then x and y are not connected directly by a single line.

Example:-

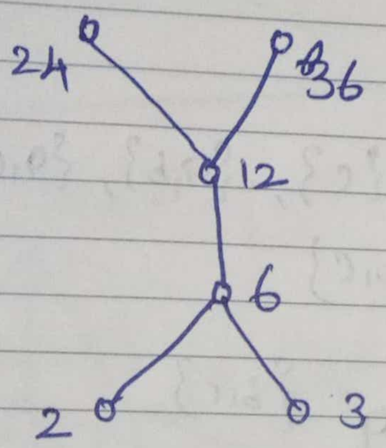
1) $P = \{1, 2, 3, 4\}$

$(P, \leq)$ "$\leq$" be the relation "less than or equal to,"

The Hasse diagram is



2) Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divided y. Draw the Hasse diagram of $(X, \leq)$



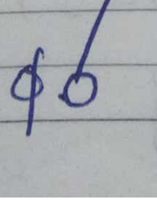
3) Let A be a given finite set and $P(A)$ its power set. Let "$\subseteq$" be the "inclusion" relation on the elements of $P(A)$. Draw the Hasse diagrams of $(P(A), \subseteq)$

(a) $A = \{a\}$ (b) $A = \{a, b\}$ (c) $= \{a, b, c\}$

(d) $A = \{a, b, c, d\}$.

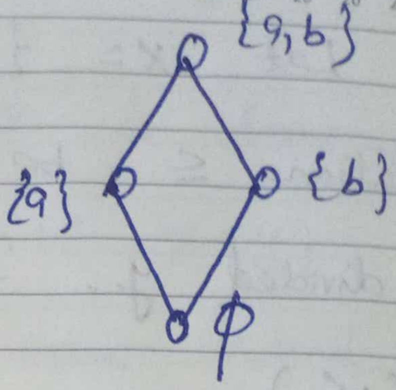
<u>Soln:-</u>

(a) $A = \{a\}$ $P(A) = \{\phi, \{a\}\}$



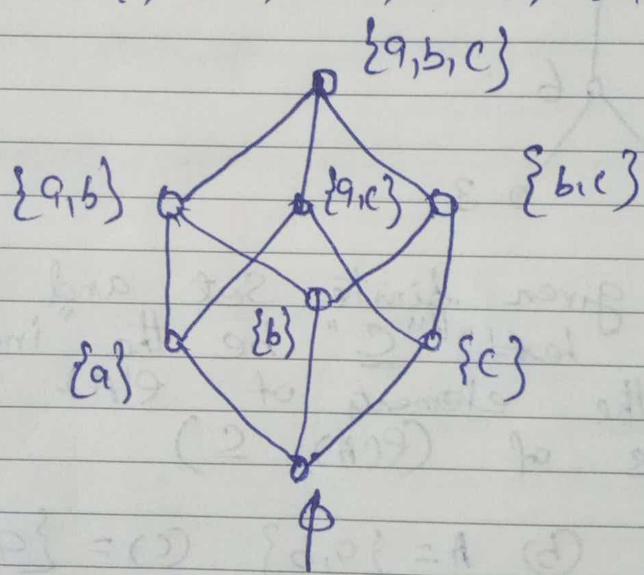
(b) $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



(c) $A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$



(d) $A = \{a, b, c, d\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}$
$\{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}$
$\{a,b,d\}, \{b,c,d\}, \{a,c,d\},$
$\{a,b,c,d\}\}$

4) Let A be the set of factors of a Particular positive integer m and let $\le$ be the relation divides, ie,

$$\le = \{(x,y) \mid x \in A \wedge y \in A \wedge (x \text{ divides } y)\}$$

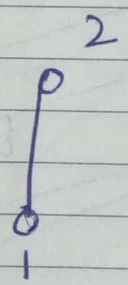
Draw Hasse diagram for

(a) $m=2$ (b) $m=6$ (c) $m=30$ (d) $m=210$

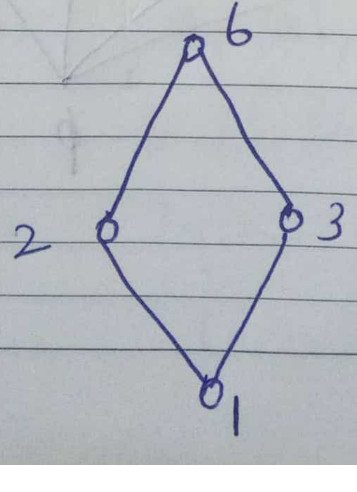
(e) $m=12$ & (f) $m=45$.

(a) $m=2$

$\le = \{1, 2\}$



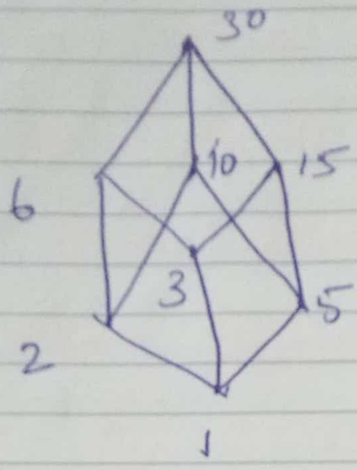
(b) $m=6$, $\le = \{1, 2, 3, 6]$



Important Notes

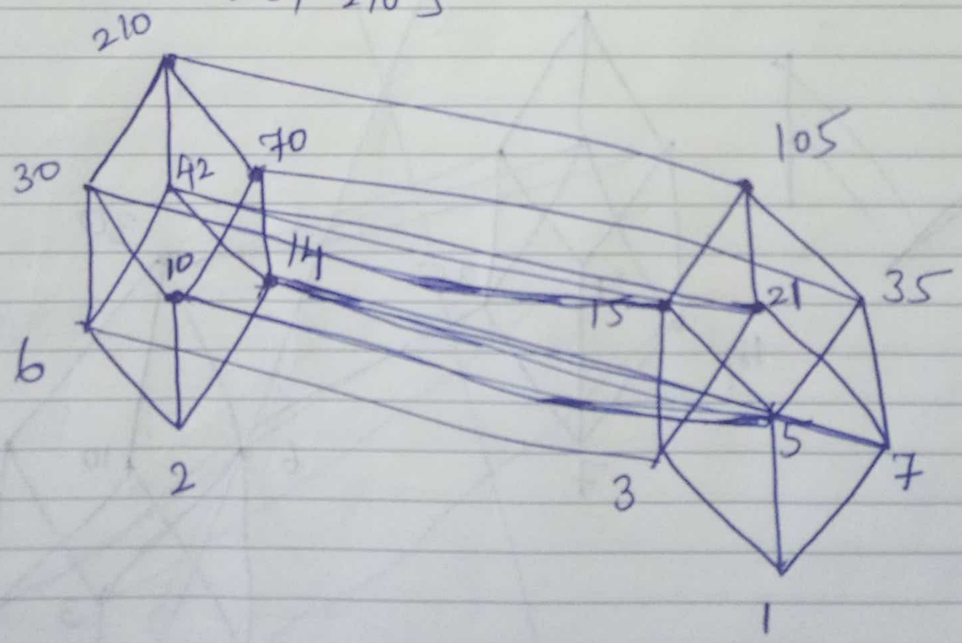
(17)

c) $m = 30$ $\leq = \{1, 2, 3, 5, 6, 10, 15, 30\}$

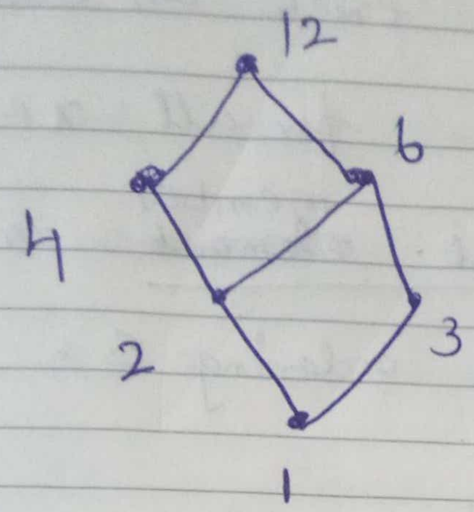


d) $m = 210$,

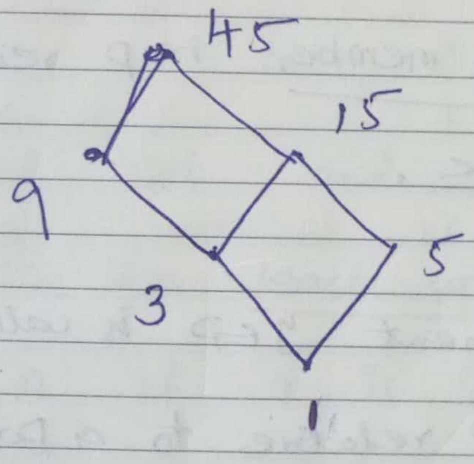
$\leq = \{1, 2, 3, 6, 5, 7, 10, 14, 15, 21, 30, 35, 42, 70,$
$105, 210\}$



e) $m = 12$ $\leq = \{1, 2, 3, 4, 6, 12\}$



f) $m = 45$ $\leq = \{1, 3, 5, 9, 15, 45\}$



8.0
9.0
10.0
11.0
12.0
1.0
2.0
3.0
4.0
5.0
6.0
Important Notes

least member:- If there exist an element $y \in P$ such that $y \leq x$ for all $x \in P$, then y is called the least member ~~element~~ in P relative to the partial ordering $\leq$.

Greatest member:- If there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called the greatest member in P relative to the partial ordering $\leq$.

Minimal member:- An element $y \in P$ is called a minimal member of P relative to a partial ordering $\leq$ if for no $x \in P$ is $x < y$

Maximal member:- An element $y \in P$ is called a maximal member of P relative to a partial ording $\leq$ if for no $x \in P$ is $y < x$.

Definition :-

Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. Any element $x \in P$ is an upper bound for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is a lower bound for A if for all $a \in A$, $x \leq a$.

Definition :-

Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. An element $x \in P$ is a least upper bound or Supremum, for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A. Similarly, the greatest lower bound or infimum for A is an element $x \in P$ such that x is a lower bound and $y \leq x$ for all lower bounds y.

Important Notes

(21)

Definition :- A Partially Ordered set is called well-ordered if every non-empty subset of it has a least member.

2.4. Functions

2.4.1. Definition and Introduction

Function

Let X and Y be any two sets. A relation f from X to Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $\langle x, y \rangle \in f$.

* The range of f is defined as

$$\{ y \,/\, \exists\, x \in X \wedge y = f(x) \}$$

Important Notes

* The range of f is denoted by R_f and $R_f \subseteq Y$.

* The domain of f is X and denoted by $D_f = X$.

Example of functions :-

1) Let $X = \{1, 5, P, \text{Jack}\}$

$Y = \{2, 5, 7, 9, \text{Jill}\}$

and $f = \{\langle 1, 2\rangle, \langle 5, 7\rangle, \langle P, 9\rangle, \langle \text{Jack}, 9\rangle\}$

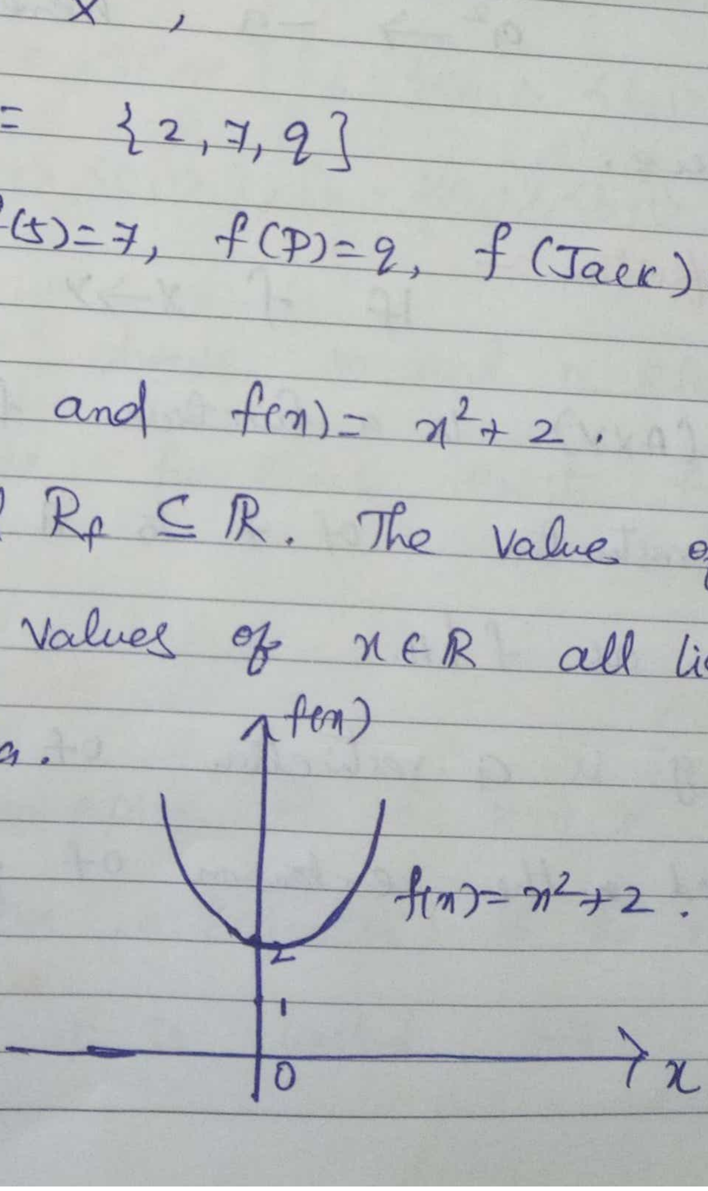
obviously $D_f = X$,

$R_f = \{2, 7, 9\}$

and $f(1) = 2$, $f(5) = 7$, $f(P) = 9$, $f(\text{Jack}) = 9$.

2) Let $X = Y = \mathbb{R}$ and $f(x) = x^2 + 2$.

$D_f = \mathbb{R}$ and $R_f \subseteq \mathbb{R}$. The value of f for different values of $x \in \mathbb{R}$ all lie on a ParaBola.



3) Let $X = Y = \mathbb{R}$ such that

$$f = \{ \langle x, x^2 \rangle \mid x \in \mathbb{R} \}$$

$$g = \{ \langle x^2, x \rangle \mid x \in \mathbb{R} \}$$

Clearly, f is a function from X to Y.

But g is not a function because

$a^2 \rightarrow a$ and

$a^2 \rightarrow -a$ here image is not unique.

Definition :- If $f : X \rightarrow Y$ and $A \subseteq X$ then $f \cap (A \times Y)$ is a function from $A \rightarrow Y$ called the restriction of f to A and is sometimes written as f / A.

If g is a restriction of f, then f is called the extension of g.

Note :- Let $X = \{a, b, c\}$, $Y = \{0, 1\}$, Then $X \times Y = \{\langle a,0\rangle, \langle a,1\rangle, \langle b,0\rangle, \langle b,1\rangle, \langle c,0\rangle, \langle c,1\rangle\}$

and there 2^6 possible subsets of $X \times Y$. Of these, only the 2^3 subsets are define a function from X to Y.

i.e, $f_0 = \{\langle a,0\rangle, \langle b,0\rangle, \langle c,0\rangle\}$, $f_4 = \{\langle a,1\rangle, \langle b,0\rangle, \langle c,0\rangle\}$

$f_1 = \{\langle a,0\rangle, \langle b,0\rangle, \langle c,1\rangle\}$, $f_5 = \{\langle a,1\rangle, \langle b,0\rangle, \langle c,1\rangle\}$

$f_2 = \{\langle a,0\rangle, \langle b,1\rangle, \langle c,0\rangle\}$, $f_6 = \{\langle a,1\rangle, \langle b,1\rangle, \langle c,0\rangle\}$

$f_3 = \{\langle a,0\rangle, \langle b,1\rangle, \langle c,1\rangle\}$, $f_7 = \{\langle a,1\rangle, \langle b,1\rangle, \langle c,1\rangle\}$

* If X and Y have m and n elements then $\underline{n^m}$ possible functions exists from X and Y.

Definition :- Onto

A mapping of $f: X \to Y$ is called onto (surjective, a surjection) if the range $R_f = Y$, otherwise it is called into.

Important Notes

~~Definition~~ One-one

A mapping $f: X \to Y$ is called one-to-one (injective or ~~injection~~ 1-1) if ~~dig~~ distinct elements of X are mapped into distinct elements of Y. In other words, f is one-to-one if,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

(or)

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ .}$$

Definition :- Bijective

A mapping $f: X \to Y$ is called one-to-one onto (bijective). If it is both one-to-one and onto. Such a mapping is also called a one-to one correspondence between X and Y.

Important Notes

2.4.2. Composition of Functions ①

Definition :- Composition of functions

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The composite relation $g \circ f$ such that

$$g \circ f = \{ \langle x, z \rangle \mid (x \in X) \wedge (z \in Z) \wedge (\exists y)(y \in Y \wedge y = f(x) \wedge z = g(y)) \}$$

is called the composition of functions or relative product of functions f and g.

* $g \circ f$ is called the left composition of g with f.

Result :- Composition of functions is associative

Proof :- Consider three functions

$f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$

Prove that, $h \circ (g \circ f) = (h \circ g) \circ f$

~~ie, to prove~~ $\cancel{(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)}$

~~i.e,~~ $\cancel{(h \circ (g \circ f))(x)}$

Important Notes

Example 1:- let $X = \{1, 2, 3\}$, $Y = \{p, q\}$

$Z = \{a, b\}$. Also let $f : X \to Y$, be

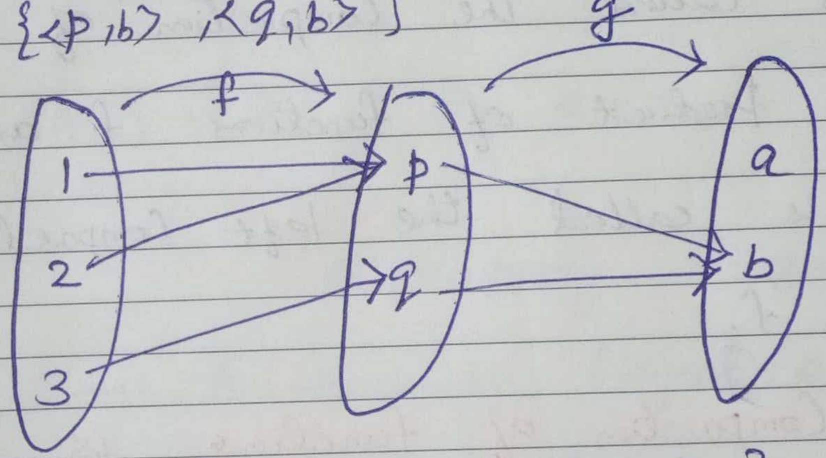
$f = \{\langle 1, p\rangle, \langle 2, p\rangle, \langle 3, q\rangle\}$ and

$g : Y \to Z$ be given by $g = \{\langle p, b\rangle \langle q, b\rangle\}$

Find $g \circ f$.

$f = \{\langle 1, p\rangle, \langle 2, p\rangle, \langle 3, q\rangle\}$

$g = \{\langle p, b\rangle, \langle q, b\rangle\}$



$\therefore$ $g \circ f = \{\langle 1, b\rangle, \langle 2, b\rangle \langle 3, b\rangle\}$.

Example 2:- let $X = \{1, 2, 3\}$, and f, g, h and s be function from X to X given by,

Important Notes

$f = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$,

$g = \{\langle 1, 2\rangle, \langle 2, 1\rangle \langle 3, 3\rangle\}$

$h = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 1\rangle\}$, ③

$S = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$

Find $f\circ g$, $g\circ f$, $f\circ h\circ g$, $S\circ g$, $g\circ S$, $S\circ S$ and $f\circ S$, $h\circ g$, $h\circ f$, $h\circ g\circ S$, also verify $(h\circ f)\circ g = h\circ(f\circ g)$.

Soln :-

(1) $f\circ g$

$(f\circ g)(1) = f(g(1)) = f(2) = 3$

$(f\circ g)(2) = f(g(2)) = f(1) = 2$

$(f\circ g)(3) = f(g(3)) = f(3) = 1$

$f\circ g = \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$

(ii) $S\circ g$

$(S\circ g)(1) = S(g(1)) = S(2) = 2$

$(S\circ g)(2) = S(g(2)) = S(1) = 1$

$(S\circ g)(3) = S(g(3)) = S(3) = 3$

$S\circ g = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$

Important Notes

Example 3:- Let $f(x) = x+2$, $g(x) = x-2$ and $h(x) = 3x$ for $x \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Find gof, fog, fof, gog, foh, hog, hof, $hofog$ and also check,

$f o(hog) = (foh) o g$.

Soln:-

(i) gof

$$(gof)(x) = g(f(x))$$
$$= g(x+2)$$
$$= (x+2)-2 = x$$

$\therefore\ gof = \{\langle x, x\rangle \mid x \in \mathbb{R}\}$

(ii) hog

$$(hog)(x) = h(g(x))$$
$$= h(x-2)$$
$$= 3(x-2) = 3x-6$$

$hog = \{\langle x, 3x-6\rangle \mid x \in \mathbb{R}\}$

Important Notes

(ii) ~~$ho(fog)$~~ $hofog$

$$(hofog)(x) = h(f(g(x)))$$
$$= h(f(x-2))$$
$$= h((x-2)+2)$$
$$= h(x)$$
$$= 3x$$

$\therefore\ hofog = \{ \langle x, 3x\rangle \,/\, x \in \mathbb{R} \}$

Remaining sums are homework.

2.4.3. Inverse Functions :-

Converse ~~Inverse~~ Relation :-

Let X and Y be any two sets, and R is a relation from X to Y, a relation $\tilde{R}$ from Y to X is called the converse of R.

for $x \in X$ and $y \in Y$ that $xRy \Leftrightarrow y\bar{R}x$

Examples:-

1. Let $X = \{1,2,3\}$, $Y = \{p,q,r\}$ and $f: X \to Y$ given by $f = \{\langle 1,p\rangle, \langle 2,q\rangle, \langle 3,q\rangle\}$

Then $\bar{f} = \{\langle p,1\rangle, \langle q,2\rangle, \langle q,3\rangle\}$

$\bar{f}$ is not a function.

Important Notes

2) Let $\mathbb{R}$ is the set of real numbers, and let $f: \mathbb{R} \to \mathbb{R}$ by

$$f = \{\langle x, x^2\rangle / x \in \mathbb{R}\}$$

Then $\bar{f} = \{\langle x^2, x\rangle / x \in \mathbb{R}\}$ is not a function.

3. Let $\mathbb{R}$ be the set of real numbers and let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f = \{\langle x, x+2\rangle / x \in \mathbb{R}\}$$

Then $\bar{f} = \{\langle x+2, x\rangle / x \in \mathbb{R}\}$ is a function from $\mathbb{R} \to \mathbb{R}$.

4. Let $X = \{0,1\}$, $Y = \{p, q, r, s\}$ and $f = \{\langle 0, p\rangle, \langle 1, r\rangle\}$ then $\bar{f} = \{\langle p, 0\rangle, \langle r, 1\rangle\}$ is a function from a subset of Y to X.

ie, $\bar{f}: \{p, r\} \to \{0, 1\}$.

Definition:- Inverse function.

Let X and Y be any two sets and if $f: X \to Y$ is one-one and onto then $\bar{f}: Y \to X$ is a function. $\bar{f}$ is written as f^{-1} and f^{-1} is called the inverse of the function f.

Important Notes

* If f^{-1} is exists then f is called invertible.

* f^{-1} is obviously one-one and onto.

Definition:- Identity Map.

A mapping $I_x : X \to X$ is called an "Identity map" if $I_x = \{\langle x,x\rangle / x \in X\}$

Theorem 2.4.1:-

If $f: X \to Y$ is inverible then

$$f^{-1} \circ f = I_x \quad \text{and} \quad f \circ f^{-1} = I_y \quad -(1)$$

Proof:- Given $f: X \to Y$ is invertible.

f is 1-1 and onto,

i.e, f^{-1} is exists and one-one, onto.

$f^{-1} : Y \to X$.

find $f^{-1} \circ f$

$(f^{-1} \circ f)(x) = f^{-1}(f(x))$

$=$

Let $x \in X$ $\exists$ $y \in Y$ $\ni$ $f(x) = y$

and f is 1-1 onto

$\therefore$ $x = f^{-1}(y)$

Now,

$$f^{-1}of : X \xrightarrow{f} Y \xrightarrow{f^{-1}} X$$

So $f^{-1}of$ is a function from X into X

and $(f^{-1}of)(x) = f^{-1}(f(x))$

$= f^{-1}(y)$ $\quad [\because f(x) = y]$

$= x$ $\quad [\because f^{-1}(y) = x]$

$\therefore$ $f^{-1}of = \{\langle x, x\rangle / x \in \mathbb{R}\} = I_X$.

Similarly, $fof^{-1} : Y \xrightarrow{f^{-1}} X \xrightarrow{f} Y$

So fof^{-1} is a function from Y to Y and

$(fof^{-1})(y) = f(f^{-1}(y)) = f(x) = y$

$\therefore$ $fof^{-1} = \{\langle y, y\rangle \mid y \in \mathbb{R}\} = I_Y$.

$\therefore f^{-1} \circ f = I_x$ and 12/7/20

$f \circ f^{-1} = I_y$.

Theorem:- 2.4.2. Let $f: X \to Y$ and $g: Y \to X$. The function g is equal to f^{-1} only if $g \circ f = I_x$ and $f \circ g = I_y$.

Proof:- Given, $f: X \to Y$ and
$g: Y \to X$, $g \circ f = I_x$ and $f \circ g = I_y$

First we ~~show~~ show that, if there is any other function $h: Y \to X$ such that
$h \circ f = I_x$ and $f \circ h = I_y$, then $h = g$.

Now, $h \circ f : X \xrightarrow{f} Y \xrightarrow{h} X$ so $h \circ f$ is a function from X to X
and $(h \circ f)(x) = h(f(x))$.

Important Notes Let $x \in X$ then $\exists\, y \in Y \ni f(x) = y$
and $y \in Y$ then $\exists\, x \in X \ni h(y) = x$

$\therefore (h \circ f)(x) = h(f(x)) = h(y) = x$

$\therefore h \circ f = I_x$

and

$f \circ h : Y \xrightarrow{h} X \xrightarrow{f} Y$, so $f \circ h$ is a function from Y to X and

$$(f \circ h)(y) = f(h(y)) = f(x) = y$$

$\therefore \quad f \circ h = I_Y.$

Now prove that $h = g$.

and we know that,

$$(h \circ f) \circ g = h \circ (f \circ g)$$

$$(h \circ f) \circ g = I_X \circ g = g$$

and $h \circ (f \circ g) = h \circ I_Y = h$

But $g = (h \circ f) \circ g = h \circ (f \circ g) = h$

$\therefore \quad h = g$

and using thm 2.4.1. we get

$$g = f^{-1}.$$

Hence the theorem is proved.

Result:- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Let $f : X \to Y$ and $g : Y \to Z$

We construct a composite function as,

$g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$

$\therefore \; g \circ f : X \to Z$

If f and g are both one-one and onto. the $g \circ f$ is also one-one and onto.

and also f^{-1}, g^{-1} and $(g \circ f)^{-1}$ are exists, and are one-one and onto.

$f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$

From these functions we can form,

$f^{-1} \circ g^{-1} : Z \to X$

Important Notes and $(g \circ f)^{-1} : Z \to X$

Both $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from Z to X.

Consider any $x \in X$ and let $y = f(x)$ and $z = g(y)$.

Thus ~~zo~~ $\langle x, z \rangle \in g \circ f$ and

$\langle z, x \rangle \in (g \circ f)^{-1}$

on the other hand

$x = f^{-1}(y)$ and $y = g^{-1}(z)$.

so that $\langle z, x \rangle \in f^{-1} \circ g^{-1}$

This is true for any $x, y,$ and z which satisfy $y = f(x)$ and $z = g(y)$.

$\therefore \quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

ie, The inverse of a composite function can be expressed in terms of the composition of the inverses in the reverse order.

Note:-

Let F_x denote the collection of all bijective functions from X onto X, so that the elements of F_x are all invertible functions. The following properties hold.

1) For any $f, g \in F_x$, $f \circ g$ and $g \circ f$ are also in F_x. This is called the closure property of the operation of composition.

2. For any $f, g, h \in F_x$

$$(f \circ g) \circ h = f \circ (g \circ h)$$

i.e, composition is associative

3. There exists a function $I_x \in F_x$ called the identity map such that for any $f \in F_x$

$$I_x \circ f = f \circ I_x = f.$$

Important Notes

4. For every $f \in F_x$, there exists an inverse function $f^{-1} \in F_x$ such that

$$f \circ f^{-1} = f^{-1} \circ f = I_x.$$

Example 1. Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$ for $x \in \mathbb{R}$ are inverses of one another.

Soln:- $f(x) = x^3$ and
$g(x) = x^{1/3}$.

$$(f \circ g)(x) = f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

$$\therefore (f \circ g)(x) = x = I_x$$

and $(g \circ f)(x) = g(f(x)) = g(x^3) = (x^3)^{1/3} = x$

$$(g \circ f)(x) = x = I_x.$$

$$\therefore f = g^{-1} \quad (\text{or}) \quad g = f^{-1}.$$

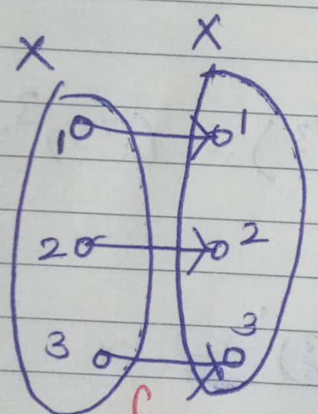
Important Notes

Example 2: Let F_x be the set of all one-to one mappings from X onto X where $X = \{1, 2, 3\}$. Find all the elements of F_x and find the inverse of each element.

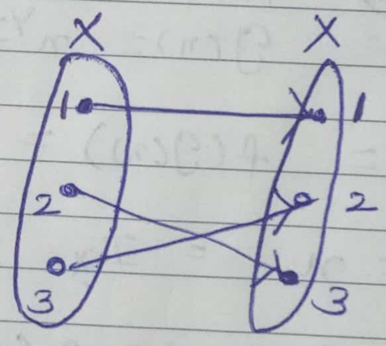
Soln:-

$$X = \{1, 2, 3\}$$

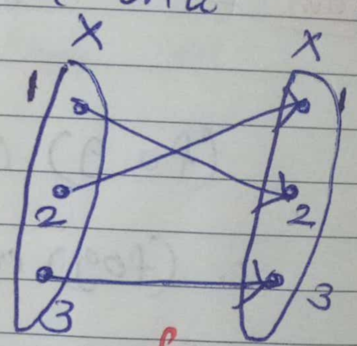
Find $F_x = \{ f : X \to X \,/\, f \text{ is one-one and onto} \}$



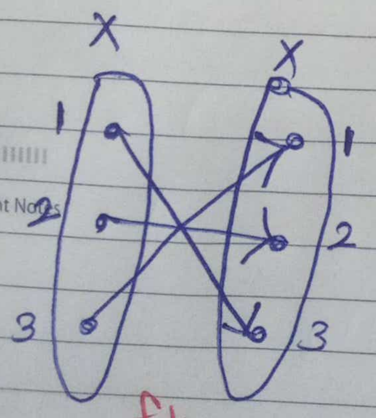
f_1



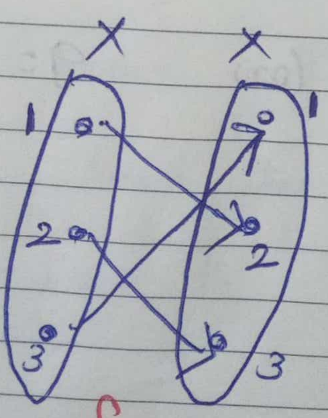
f_2



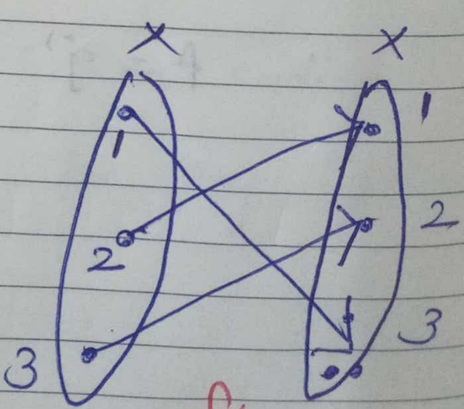
f_3



f_4



f_5



f_6

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_6	f_5	f_4	f_3
f_3	f_3					
f_4	f_4					
f_5	f_5					
f_6	f_6					

$\underline{f_1 \circ f_1}$ $\quad (f_1 \circ f_1)(1) = f_1(f_1(1)) = f_1(1) = 1$

$(f_1 \circ f_1)(2) = f_1(f_1(2)) = f_1(2) = 2$

$(f_1 \circ f_1)(3) = f_1(f_1(3)) = f_1(3) = 3$

$\therefore \quad f_1 \circ f_1 = f_1$

~~S.t~~ Simarly find $f_1 \circ f_i = f_i \quad (i = 2, 3, 4, 5, 6)$

and $f_i \circ f_1 = f_i \quad (i = 2, 3, 4, 5, 6)$

$\underline{f_2 \circ f_2}$: $(f_2 \circ f_2)(1) = f_2(f_2(1)) = f_2(1) = 1$

$(f_2 \circ f_2)(2) = f_2(f_2(2)) = f_2(3) = 2$

$(f_2 \circ f_2)(3) = f_2(f_2(3)) = f_2(2) = 3$

$\therefore \quad f_2 \circ f_2 = f_1$

Important Notes

$f_2 \circ f_3$: $(f_2 \circ f_3)(1) = f_2(f_3(1)) = f_2(2) = 3$

$(f_2 \circ f_3)(2) = f_2(f_3(2)) = f_2(1) = 1$

$(f_2 \circ f_3)(3) = f_2(f_3(3)) = f_2(3) = 2$

$f_2 \circ f_3 = f_6$

$f_2 \circ f_4$: $(f_2 \circ f_4)(1) = f_2(f_4(1)) = f_2(3) = 2$

$(f_2 \circ f_4)(2) = f_2(f_4(2)) = f_2(2) = 3$

$(f_2 \circ f_4)(3) = f_2(f_4(3)) = f_2(1) = 1$

$\therefore f_2 \circ f_4 = f_5$

$f_2 \circ f_5$: $(f_2 \circ f_5)(1) = f_2(f_5(1)) = f_2(2) = 3$

$(f_2 \circ f_5)(2) = f_2(f_5(2)) = f_2(3) = 2$

$(f_2 \circ f_5)(3) = f_2(f_5(3)) = f_2(1) = 1$

$\therefore f_2 \circ f_5 = f_4$

$f_2 \circ f_6$: $(f_2 \circ f_6)(1) = f_2(f_6(1)) = f_2(3) = 2$

Important Notes

$(f_2 \circ f_6)(2) = f_2(f_6(2)) = f_2(1) = 1$

$(f_2 \circ f_6)(3) = f_2(f_6(3)) = f_2(2) = 3$

$\therefore f_2 \circ f_6 = f_3$

Finish the remaining Values of the table.

Here, Identity map is "f_1" function

From the table,

$$f_1 \circ f_1 = f_1,$$

$$f_2 \circ f_2 = f_1$$

$$f_3 \circ f_3 = f_1$$

$$f_4 \circ f_4 = f_1$$

$$f_5 \circ f_6 = f_1 \quad \text{and}$$

$$f_6 \circ f_5 = f_1$$

The inverses of the functions are,

$\therefore$ $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$,

$f_5^{-1} = f_6$ and $f_6^{-1} = f_5$.

①

2.5 Natural Numbers

2.5.1. Peano Axioms and Mathematical Induction

Peano Axiom:-

(1) $0 \in \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

(2) If $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$ where

$n^+ = n \cup \{n\}$

(3) If a subset $S \subseteq \mathbb{N}$ possesses the properties
(a) $0 \in S$ and
(b) If $n \in S$, then $n^+ \in S$
then $S = \mathbb{N}$.

Note :- * property (3) is known as the "minimality property".

* n^+ means $n+1$.

Important Notes

②

Principle of Mathematical Induction

If $P(n)$ is any property defined over the set of Natural numbers and

(a) $P(0)$ is true

(b) If $P(m) \Rightarrow P(m^+)$ for any $m \in \mathbb{N}$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Example 1: Show that $n < 2^n$.

Soln:-

Let $P(n) : n < 2^n$

(a) For $n = 0$, $P(0) : 0 < 2^0 = 1$

$\therefore P(0)$ is true

For $n = 1$, $P(1) : 1 < 2^1$.

i.e, $1 < 2$

$\therefore P(1)$ is true.

(b) For some arbitrary choice of $m \in \mathbb{N}$ assume that $P(m)$ holds and prove that $P(m+1)$ is satisfied.

i.e, $P(m): \quad m < 2^m$

$$\Rightarrow \quad m+1 < 2^m + 1$$

$$< 2^m + 2^m$$

$$= 2^m(1+1)$$

$$= 2 \cdot 2^m$$

$$= 2^{m+1}$$

i.e, $m+1 < 2^{m+1}$

$\therefore$ $P(m+1)$ is hold.

$\therefore$ $P(m) \Rightarrow P(m+1)$.

Hence from the mathematical of Induction,

$P(n)$ is true for all $n \in \mathbb{N}$.

Example 2:- Show that $2^n < n!$ for $n \geq 4$.

Soln:- Let $P(n): 2^n < n!$ for $n \geq 4$.

$P(1): 2 \nless 1,$

$P(2): 2^2 \nless 2$

$P(3): 2^3 \nless 6$

(4)

$\therefore$ $P(1)$, $P(2)$, $P(3)$ is not True.

Now $P(4)$: $2^4 < 4!$

i.e, $P(4)$: $16 < 24$

So that $P(4)$ holds.

Assume that $P(m)$ holds for any $m > 4$.

and so,

$$2^m < m!$$

Multiply both sides by 2 we get,

$$2 * 2^m < 2(m!)$$

$$< (m+1) * (m!)$$

$$= (m+1)!$$

i.e, $2^{m+1} < (m+1)!$

i.e, $P(m+1)$: $2^{m+1} < (m+1)!$ holds.

Hence $P(n)$ holds for all $n \in \mathbb{N}$ and $n \geq 4$.

Example : 3 Show that

$$B \cup \left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} (B \cup A_i)$$

Soln:-

Let $P(n): B \cup \left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} (B \cup A_i)$

For $n=2$,

$$B \cup (A_1 \cap A_2) = (B \cup A_1) \cap (B \cup A_2) \quad -(1)$$

which follows from the distributive law of Union and intersection.

Now, Assume that $P(m)$ holds for any m, So that

$$B \cup \left(\bigcap_{i=1}^{m} A_i\right) = \bigcap_{i=1}^{m} (B \cup A_i) \quad -(2)$$

Now, $B \cup \left(\bigcap_{i=1}^{m+1} A_i\right) = B \cup \left(\bigcap_{i=1}^{m} A_i \cap A_{m+1}\right)$

Important Notes

$$= \left(B \cup \left(\bigcap_{i=1}^{m} A_i\right)\right) \cap (B \cup A_{m+1})$$

[∵ Using (1)]

$$B \cup \left(\bigcap_{i=1}^{m+1} A_i \right) = \bigcap_{i=1}^{m} (B \cup A_i) \cap (B \cup A_{m+1})$$

[∵ using ②]

$$B \cup \left(\bigcap_{i=1}^{m+1} A_i \right) = \bigcap_{i=1}^{m+1} (B \cup A_i)$$

∴ $P(m+1)$ is holds.

Hence $P(n)$ is true for all n.

Example 4:- Show that n^3+2n is divisible by 3.

soln:- Let $p(n)$: n^3+2n is divisible by 3.

Now $P(0) = 0$ is divisible by 3.

∴ $P(0)$ is true.

Let us assume ~~for~~ for any m, $P(m)$ is true

i.e, m^3+2m is divisible by 3.

Now $(m+1)^3 + 2(m+1) = m^3 + 3m^2 + 3m + 1 + 2m + 2$

$= m^3 + 2m + 3(m^2+m+1)$

⑦

Since m^3+2m is divisible by 3, and

$3(m^2+m+1)$ is also divisible by 3,

$\therefore$ $(m+1)^3 + 2(m+1)$ is divisible by 3.

i.e, $P(m+1)$ is hold.

Hence $P(n)$ is true for all $n \in \mathbb{N}$.

Home work

1. Show that

$$S(n) = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

2. Prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

3. Show that,

$$2+2^2+2^3+\cdots+2^n = 2^{n+1}-2.$$

Important Notes

4. Show that n^3+2n is divisible by n.

An Inductive definition of a Property or a set P:

1. Given a finite set A whose elements have the Property P.
2. The elements of a set B, all of which are constructed from A, Satisfy the property P.
3. The elements constructed as in (1) and (2) are the only elements satisfying the property P.

Example 5:- Find the set given by the following definition.

1) $3 \in P$

2) For $x, y \in P$, $x+y \in P$

3) only those elements obtained from step (1) and step (2) are in P.

(9)

Soln:-

1. $3 \in P$ and $3 \in P$ (by ①)

2. $3+3 = 6 \in P$ (by ②)

again $3 \in P$, $6 \in P \Rightarrow 9 \in P$ (by ②)

and $6 \in P$, $6 \in P \Rightarrow 12 \in P$

Following in this similar way we get,

$P = \{3, 6, 9, 12, 15, \ldots\}$.

i.e, The set P consists of positive integers which are multiples of 3.

Example 6:- Give an inductive definition of the set, $P = \{2, 3, 4, \ldots\} = \mathbb{N} - \{0, 1\}$

Soln:-

1. $2 \in P$ and $3 \in P$
2. If $x, y \in P$ then $x+y \in P$.
3. Only those elements obtained from steps 1. and 2 are in P.