UNIT - IV

The transformation $w = z + 1$.

Let $w = u + iv$, $z = x + iy$.

$\therefore u + iv = (x+1) + iy$

$u = x + 1 \quad v = y$.

Here every point in the z-plane is mapped into one unit right in the w-plane.

This mapping is a <u>translation</u>.

<u>$w = iz$</u>

$\cancel{u + iv = i(x + iy)}$.

Let $z = re^{i\theta}$, since $i = e^{i\pi/2}$

$$w = re^{i(\theta + \pi/2)}$$

This mapping rotates ~~ea~~ the radius vector of each non zero z through a right angle about the origin in the counterclockwise direction.

This mapping is a <u>rotation</u>.

$w = \bar{z}$

$w = \bar{z} = x - iy.$

This mapping transforms each point $z = x+iy$ into its reflection in the real axis.

This mapping is a reflection.

Explain the transformation $w = z^2$

Let $w = u+iv$ and $z = x+iy$.

$\therefore w = z^2$ implies,

$u = x^2 - y^2, \quad v = 2xy$ ——— (1).

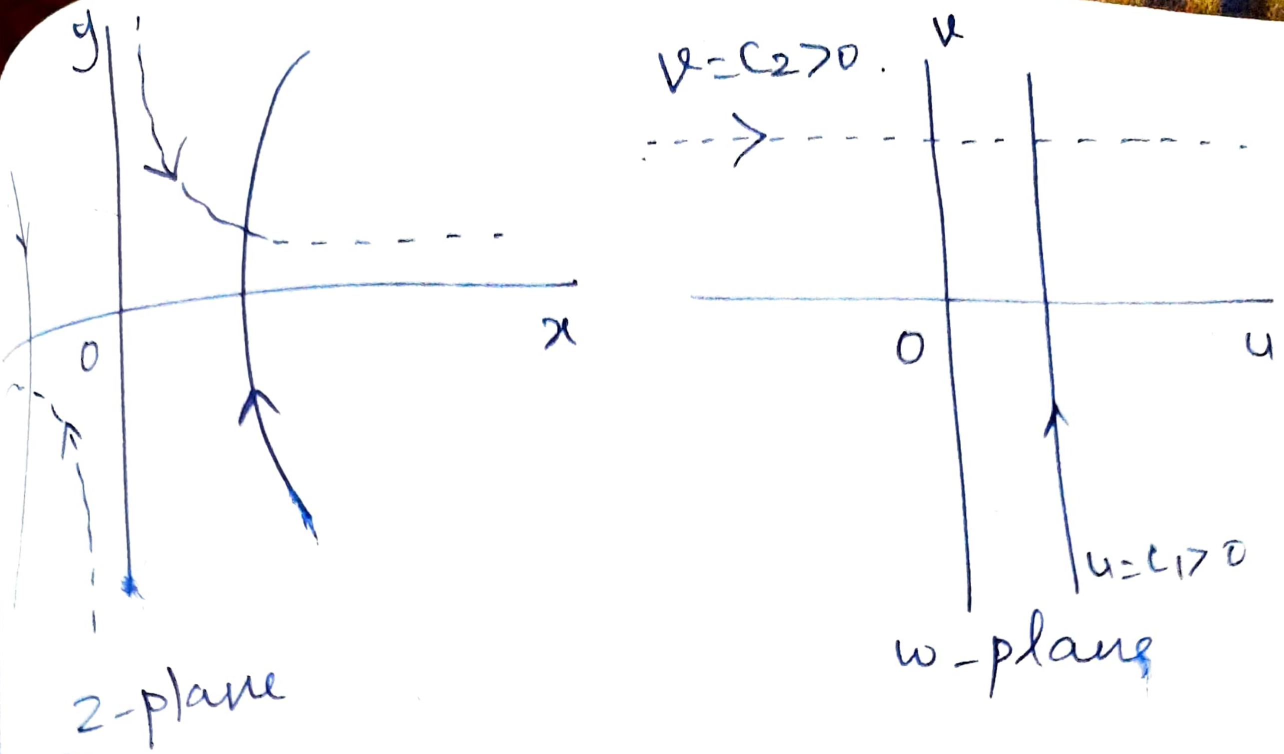
~~That~~ The transformation $w = z^2$ maps the rectangular hyperbolas into the straight lines.

Consider the hyperbola

$x^2 - y^2 = c_1 \ (c_1 > 0)$ in the xy-plane

——— (2)

This is mapped into $u = c_1$ ——— (3).



If the point (x, y) lies on the right-hand side of branch, then ① implies,

$$u = c_1 \quad ; \quad v = 2y\sqrt{y^2 + c_1} \qquad \text{——} \quad ④$$

$$-\infty < y < \infty .$$

~~the image of (x,y) moves upward~~

Hence if (x,y) traces out the branch the upward direction, its image moves upward in the line $u = c_1$.

If the point (x, y) lies on the left-hand branch, then ① implies

$$u = c_1 \quad , \quad v = -2y\sqrt{y^2 + c_1} \quad , \quad -\infty < y < \infty \qquad ⑤$$

If the point (x,y) traces out the branch in the downward direction, its image moves up the entire line $u = c_1$.

On the other hand, ~~each~~ ~~branch~~ consider the hyperbola

$$2xy = C_2 \quad , C_2 > 0 \quad \text{———} \quad (6)$$ in the xy-plane.

~~From~~ In the view of (1), each branch of (6) is mapped into $v = C_2$ in the uv-plane.

Suppose that (x,y) lies in the first quadrant.

Then, since $y = \dfrac{C_2}{2x}$,

(1) $\Rightarrow$ $\quad u = x^2 - \dfrac{C_2^2}{4x^2} \quad , \quad v = C_2 \quad , \quad 0 < x < \infty$ $\quad$ ——— (7)

Observe that $\displaystyle\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty$.

$$\lim_{x \to \infty} u = \infty .$$

and

Hence if (x,y) travels down the entire upper branch of ③, its image moves to the right along the line $v = c_2$.

Suppose that (x,y) lies in the ~~4th~~ 3rd quadrant, then

since $x = \dfrac{c_2}{2y}$,

$$① \Rightarrow \quad u = \frac{c_2^2}{4y^2} - y^2, \quad v = c_2, \quad -\infty < y < 0 \qquad (8)$$

Observe that

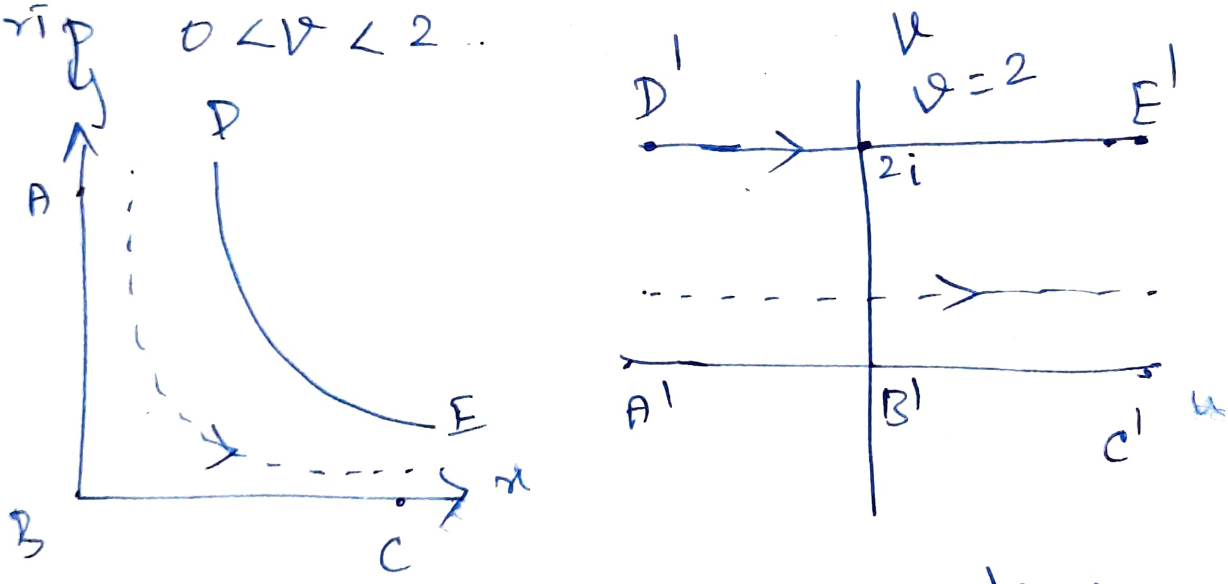
$$\lim_{y \to -\infty} u = -\infty \quad \text{and} \quad \lim_{\substack{y \to 0 \\ y<0}} u = \infty$$

Hence if (x,y) is moving upward on the lower branch, its image moves right along the entire line $v = c_2$.

The transformation $w = z^2$, maps, the closed region $x \geq 0, y \geq 0, xy \leq 1$ is mapped into the closed strip $0 \leq v \leq 2$.

For all values of c between 0 and 2, the upper branches fill out the domain $x > 0, y > 0, xy < 1$.

This domain is mapped onto the horizontal strip $0 < v < 2$.



z-plane w-plane

In the view of ①, the image of the point $(0, y)$ in the z-plane is $(-y^2, 0)$ in the w-plane.

If $(0,y)$ travels downwards to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w-plane.

Since the image of $(x,0)$ is $(x^2,0)$, if $(x,0)$ moves to the right from the origin along x axis, its image moves ~~from~~ to the right from the origin along u-axis.

Also the image of the upper branch of hyperbola $xy=1$, is the line $v=2$.

Hence the closed region $x \geq 0, y \geq 0, xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$ as shown in figure.

The transformation $w = z^2$ maps the First quadrant ~~plane~~, ~~upper half~~ $r \geq 0, 0 \leq \theta \leq \pi/2$ onto the entire upper half ~~w~~-plane

~~$w = z^2 \Rightarrow w = r^2 e^{i2\theta}$, where $z = re^{i\theta}$.~~

Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$.

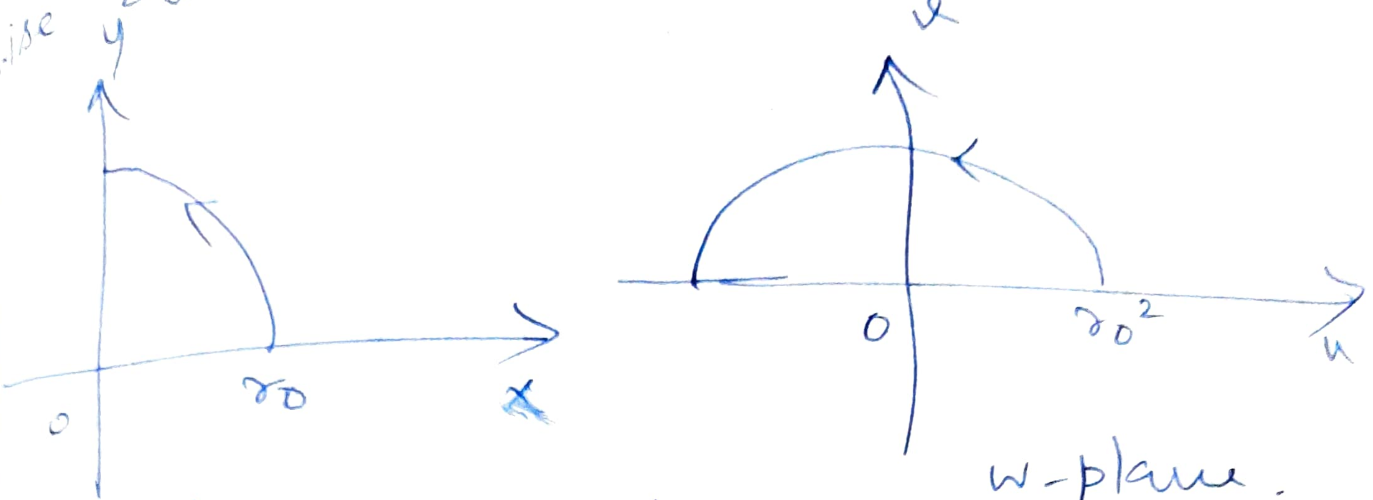
Then $w = z^2 \Rightarrow \rho e^{i\phi} = r^2 e^{i2\theta}$.

$\Rightarrow \rho = r^2$ and $\phi = 2\theta$. ——— (1)

Consider the points $z = r_0 e^{i\theta}$ on ~~the~~ a circle ~~$z = r_0 e^{i\theta}$~~ $r = r_0$.

These points are mapped into $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$.

If a point on the circle $r = r_0$, moves from ~~x~~ positive x-axis to positive y axis in the counterclockwise direction its image moves from positive u-axis to negative v-axis in the counterclockwise direction as shown in figure.



If we take all the positive values of r_0, corresponding the arcs fill out the first quadrant in the z-plane and upper half in the w-plane.

Thus, $w = z^2$ is a one to one mapping which maps the first quadrant $r \geq 0$, $0 \leq \theta \leq \pi/2$ in the z-plane onto the upper half plane $\rho \geq 0$, $0 \leq \phi \leq \pi$ in the w-plane.

The point $z = 0$, is mapped onto $w = 0$.

Explain the transformation $w = e^z$

Let $z = x+iy$ & $w = \rho e^{i\phi}$

Then $w = e^z = e^{x+iy} = e^x . e^{iy}$

Thus $\rho e^{i\phi} = e^{x} e^{iy}$

$\Rightarrow \rho = e^x$ and $\phi = y$ ——— (1)

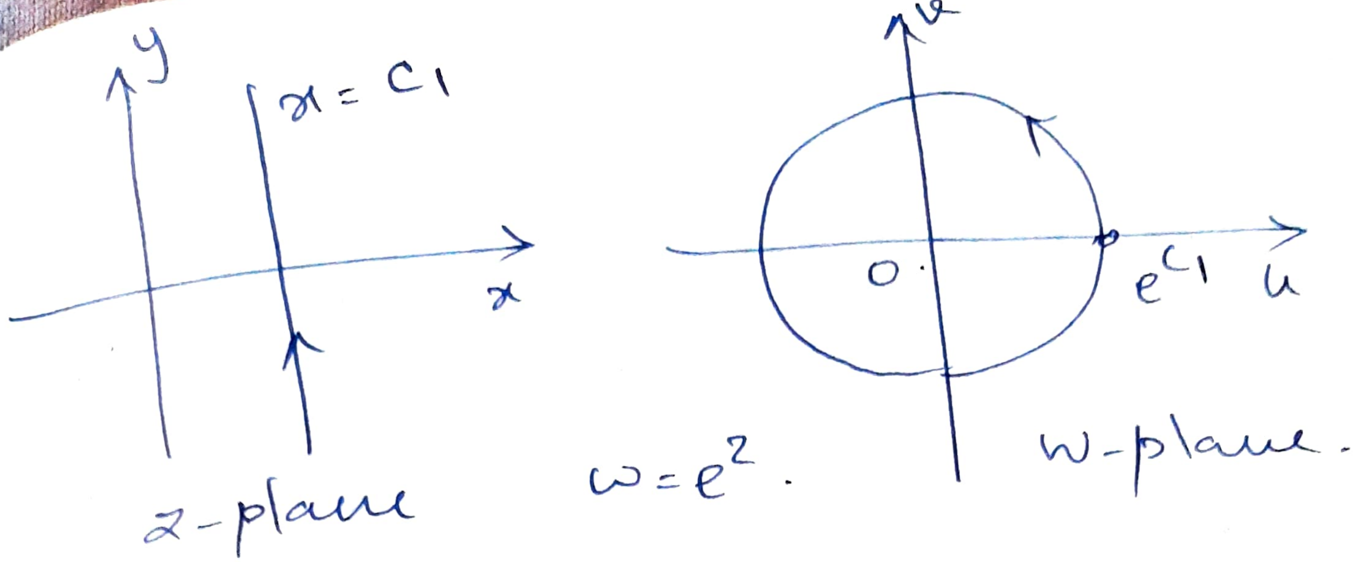
The transformation $w = e^z$ maps Vertical and horizontal lines segments onto portions of circles and rays respectively

Consider a vertical line $x = c_1$ in the z-plane.

Then (1) $\Rightarrow$ $\rho = e^{c_1}$ and $\phi = y$ in the w-plane

$\therefore$ The vertical line $x = c_1$ is mapped into the circle $\rho = e^{c_1}$.

If z moves up the line $x = c_1$, then the image moves counterclockwise around the circle $\rho = e^{c_1}$. as shown in figure.



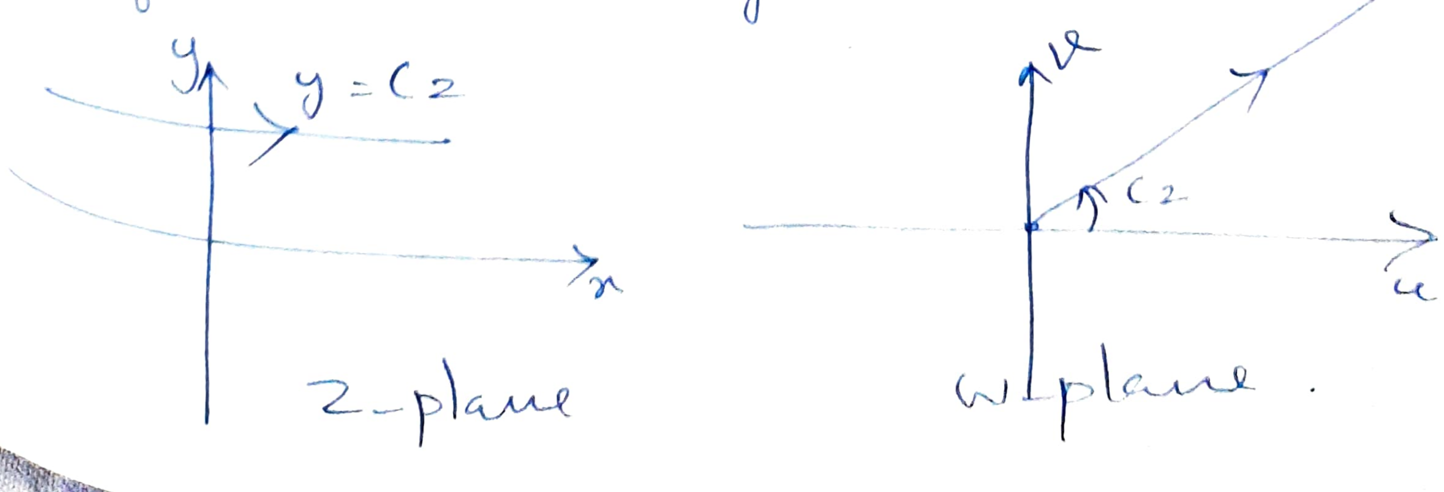
Consider a horizontal line $y = c_2$.

Then ① $\Rightarrow$ $\phi = c_2$.

$\therefore$ The image of $z = (x, c_2)$ has polar coordinates $\rho = e^x$, $\phi = c_2$.

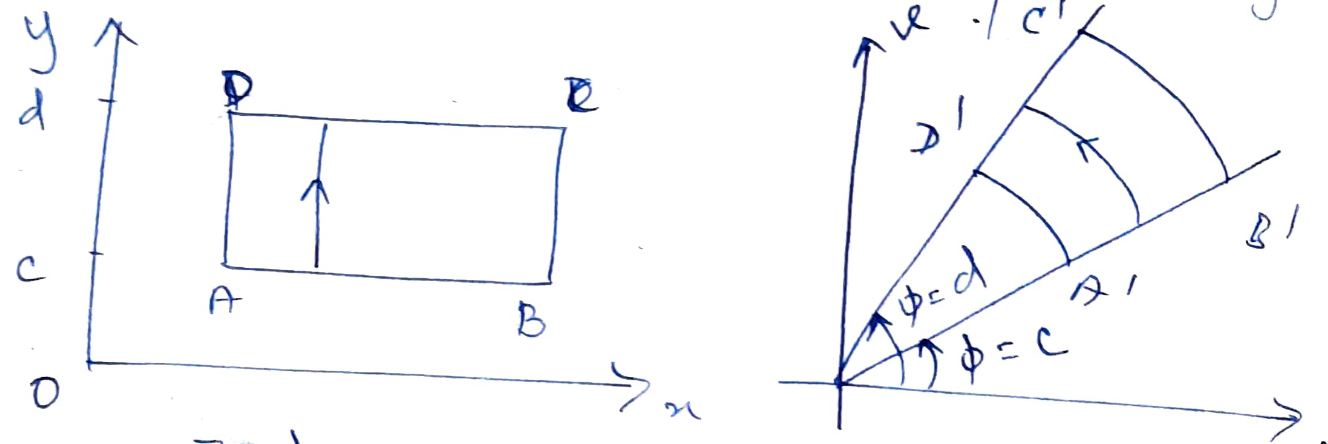
hence $y = c_2$ is mapped into a 1-1 manner onto the ray $\phi = c_2$.

If z moves along the line $y = c_2$, from left to right, its image moves along ~~outward~~ along the entire ray $\phi = c_2$.



(ii) $w = e^z$ maps the rectangular strip $a \le x \le b$, $c \le y \le d$ onto the region $e^a \le \rho \le e^b$, $c \le \phi \le d$.

From (1), the line segment AD and BC are mapped onto the arcs $\rho = e^a$, $c \le \phi \le d$ (A'D') and $\rho = e^b$, $c \le \phi \le d$ (B'C') respectively.

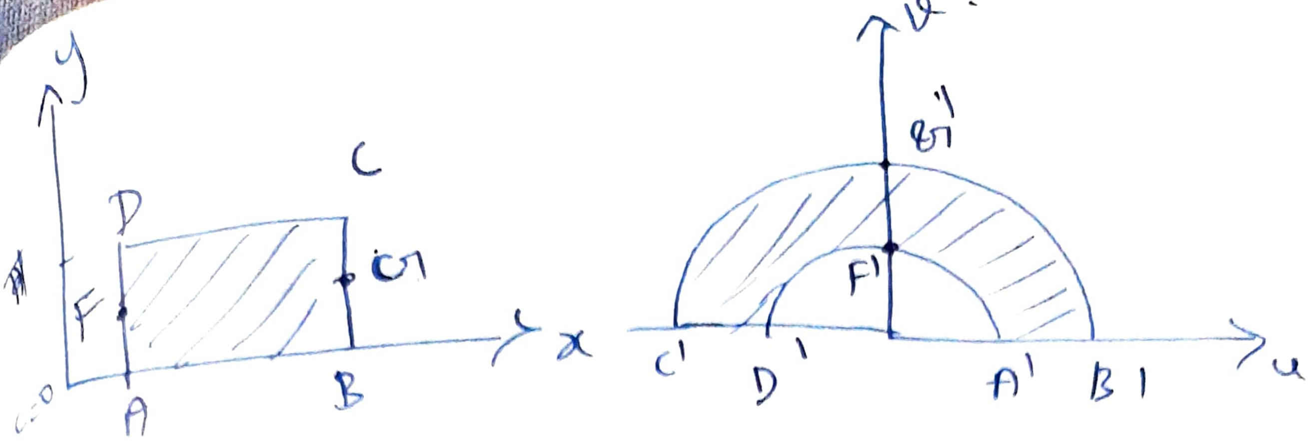


z-plane $\qquad w = e^z \qquad$ w-plane.

This mapping is one-one if $d - c < 2\pi$.

In particular, if $c = 0$ and $d = \pi$ then $0 \le \phi \le \pi$,

Then the rectangular region is mapped onto half of a circular ring as shown in the following figure.

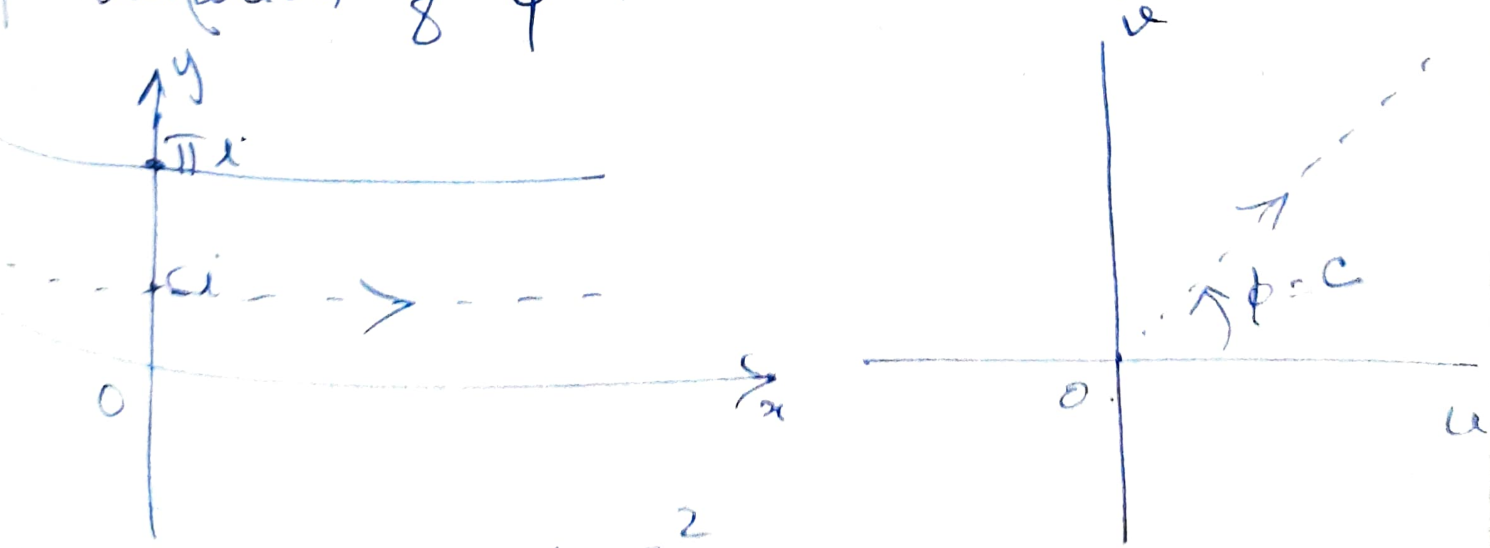


The transformation $w = e^z$ maps the infinite strip $0 \le y \le \pi$ into the upper half plane $v \geq 0$.

From (1), $y = c \Rightarrow \phi = c$.

i) The horizontal line $y = c$ is mapped to a ray $\phi = c$ from the origin.

If c increases from 0 to π, the angle of inclination of ϕ increases from $\phi = 0$ to $\phi = \pi$.



$w = e^z$.

Hence the strip $0 \le y \le \pi$ is mapped onto $v \geq 0$.

Explain the transformation $w = 1/z$.

$$w = 1/z \quad \text{———— (1)}.$$

Since $z\bar{z} = |z|^2$, the mapping can be described by means of the successive transformations,

$$Z = \frac{z}{|z|^2}, \quad \cancel{w = \bar{Z}. \text{———— (2)}}. \quad \text{———— (2)} \quad \text{and}$$

~~consider~~ $\cancel{|Z| = \frac{|z|}{|z|^2} = \frac{1}{|z|}}$

$$w = \bar{Z} \quad \text{———— (3)}.$$

$(2) \cancel{\Rightarrow} \cancel{|Z| = \frac{|z|}{|z|^2} = \frac{1}{|z|}}.$ (2)

The transformation is an inversion with respect to the unit circle $|z| = 1$.

Let $z = x + iy \neq 0$.

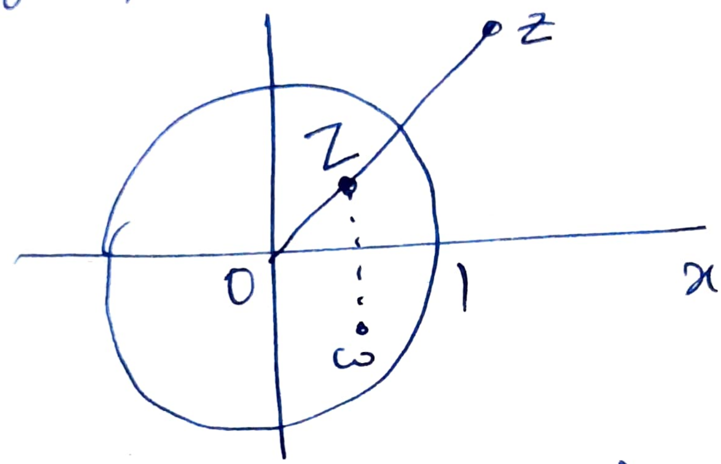
$$Z = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

$$|Z| = \sqrt{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2} = \frac{1}{|z|}$$

and $\arg Z = \tan^{-1}\left(\frac{y}{\cancel{x^2+y^2}} \times \frac{\cancel{x^2+y^2}}{x}\right)$

$= \arg z.$

Thus the points exterior to the circle $|z|=1$, are mapped onto the nonzero points interior to it. ~~y~~ and conversely.



Any point on the circle $|z|=1$, is mapped onto itself.

~~Ba~~

The transformation ③ is simply a reflection in the real axis.

$w = 1/z$ transforms circles and lines into circles and lines.

when $w = u + iv$ and $z = x + iy$ in $w = 1/z$ implies,

$$u + iv = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

$$\therefore u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2} \quad \text{---} (1)$$

Also since $z = \frac{1}{w}$, $x + iy = \frac{u - iv}{u^2 + v^2}$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2} \quad \text{---} (2)$$

when A, B, C & D are real numbers satisfying the condition $B^2 + C^2 > 4AD$, the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad \text{---} (3)$$

represents a circle if $A \neq 0$, and it represents a line if $A = 0$.

(2)(I) From (2), (3) implies is transformed into the equation,

$$A\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right) + B\frac{u}{(u^2+v^2)} - \frac{Cv}{u^2+v^2} + D = 0$$

(ii) ~~$A(u^2+v^2)$~~

$$D(u^2+v^2) + Bu - Cv + A = 0 \quad \text{--- (4)}$$

which also represents a circle or line.

Hence,

(a) ~~a circle not passing through the origin in the z-plane (A ≠ 0 & D ≠ 0), is transformed~~

(a) If $A \neq 0$ & $D \neq 0$, (3) represents a circle not passing through the origin in the z-plane.

~~Then (4) implies, represents~~

If $A \neq 0, D \neq 0$, (4) represents a circle not passing through the origin in the w-plane.

∴ A circle not passing through the origin in the z-plane is mapped into a circle not passing through the origin in the w-plane.

(b) when $A \neq 0$, and $D = 0$,

A circle passing through the origin is mapped into a line not passing through the origin.

(c) If $A = 0$ but $D \neq 0$.
Then a line not passing through the origin is transformed into a circle through the origin in the w-plane.

(d) If $A = 0$, $D = 0$,

A line through the origin in the z-plane is transformed into a line through the origin in the w-plane.

LINEAR FRACTIONAL TRANSFORMATIONS

The transformation

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

where a, b, c, d are complex constants, is called a linear fractional transformation or Mobius transformation or bilinear transformation.

Bilinear transformation maps circles and lines into circles and lines.

Consider the bilinear transformation

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

Case (i) Suppose $c = 0$.

Then $w = \frac{az+b}{d} = a/d\, z + b/d$, $ad \neq 0$

This is a non constant linear function

This transforms circles and lines into circles and lines.

Case (ii) when $c \neq 0$,

Then $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$

implies,

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}, \quad ad-bc \neq 0$$

$$\left[\begin{array}{l} cz+d \,\overline{\big)\, az+b \,}\ \frac{a}{c} \\ \quad\ \ az + \frac{ad}{c} \\ \quad\ \ \overline{\quad b - \frac{ad}{c}} \end{array} \qquad az+b = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} \right]$$

~~Let $w_1 = cz+d$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc-ad}{c} \cdot w_2$~~

~~$w = \frac{a}{c} + w_3$.~~

Let $Z = cz+d$, $W = \dfrac{1}{Z}$, $w = \dfrac{a}{c} + \dfrac{bc-ad}{c} \cdot W$

$ad - bc \neq 0$

Hence if $c \neq 0$, a bilinear transformation is a composition of Translation, inversion and, Rotation and Translation.

These transformations maps circles and lines into circles and lines.

Consider à bilinear transformation

$$T(z) = w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \text{ ———— (1)}$$

$$\Rightarrow \quad czw + dw = az + b$$

$$\Rightarrow \quad z(cw-a) = b - dw$$

$$\Rightarrow \quad z = \frac{-dw+b}{cw-a}, \quad ad-bc\neq 0.$$

$$\text{Then } T^{-1}(w) = z = \frac{-dw+b}{cw-a}, \quad ad-bc\neq 0 \text{ ———— (2)}$$

T^{-1} is also a linear fractional transformation

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when $c=0$, $T(\infty) = \infty$

when $c\neq 0$, $T(\infty) = a/c$ and

$T(-d/c) = \infty$.

when $c=0$, $T^{-1}(\infty) = \infty$.

when $c\neq 0$, $T^{-1}(a/c) = \infty$ and $T^{-1}(\infty) = -\frac{d}{c}$

Find the bilinear transformation which maps z_1, z_2, z_3 ~~ende~~ in the finite z-plane onto distinct points w_1, w_2 and w_3 respectively.

If the required bilinear transformation is

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

Then $$w_1 = \frac{az_1+b}{cz_1+d}, \quad w_2 = \frac{az_2+b}{cz_2+d},$$

$$w_3 = \frac{az_3+b}{cz_3+d}$$

$\therefore$ $$w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}$$

$$= \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

$\Rightarrow (cz+d)((z_1+d)(w-w_1) = (ad-bc)(z-z_1)$

IIIly $((z_2+d)((z_3+d)(w_2-w_3) = (ad-bc)(z_2-z_3)$

$((z+d)((z_3+d)(w-w_3) = (ad-bc)(z-z_3)$

$(cz_2+d)((z_1+d)(w_2-w_1) = (ad-bc)(z_2-z_1)$

$\Rightarrow$ $(cz+d)(cz_1+d)(w-w_1)$.
$(cz_2+d)(cz_3+d)(w_2-w_3)$ =

$(ad-bc)(z-z_1)(ad-bc)(z_2-z_3)$

&

$(cz+d)(cz_3+d)(w-w_3)(cz_2+d)(cz_1+d)$
(w_2-w_1) =

$(ad-bc)(ad-bc)(z-z_3)(z_2-z_1)$.

$$\Rightarrow \boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$ ①.

Problems :-

① Find the linear fractional transformation, which maps
$z_1=2, z_2=i, z_3=-2$ onto the points
$w_1=1, w_2=i, w_3=-1$.

Soln :-

The B.T which maps z_1, z_2, z_3 onto w_1, w_2, w_3 is

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\Rightarrow \frac{(z-2)(i+2)}{(z+2)(i-2)} = \frac{(w-1)(i+1)}{(w+1)(i-1)}$$

$$\Rightarrow (z-2)(w+1)(-1+2i-i-2) =$$
$$(w-1)(z+2)(-1+i-2i-2)$$

$$\Rightarrow (zw+z-2w-2)(i-3) =$$
$$\cancel{w-1)}\ (wz+2w-z-2)(-i-3)$$

$$\Rightarrow [w(z-2)+(z-2)]\ i-3 =$$
$$-[w(z+2)-(z+2)](i+3)$$

$$\Rightarrow \cancel{w(+1)}$$

$$\Rightarrow 18zw+4w-2z-36+12iw+12i$$
$$-6izw-6iz=0$$

$$\Rightarrow w(18z+4+12i-6iz) = 2z+3i-12i+6i$$

$$\Rightarrow \frac{(w-1)}{w+1} = \frac{(z-2)(i+2)(i-1)}{(z+2)(i-2)(i+1)}$$

$$\frac{w-1}{w+1} = \frac{(z-2)(-1+2i-i-2)}{(z+2)(-1-2i+i-2)}$$

$$\frac{w-1}{w+1} = \frac{(z-2)(i-3)}{(z+2)(-i-3)}$$

$$\frac{w-1}{w+1} = \frac{-(z-2)(i-3)^2}{(z+2)\,10}$$

$$\Rightarrow \quad = \frac{-(z-2)\overset{(8-6i)}{(-1+9-6i)}}{10(z+2)}$$

$$10(w-1)(z+2) = (w+1)(2-z)(8-6i)$$

$$\Rightarrow 10\left[wz+2w-z-2\right] = (8-6i)(2w+2-zw-z)$$

$$\Rightarrow \cancel{10z+20-16+8z+12i = 6iz}$$

$$\Rightarrow \overset{18zw}{\cancel{10wz}} + \overset{4w}{\cancel{20w}} - \overset{2z}{\cancel{10z}} - \overset{-36}{\cancel{20}} = \cancel{16w} + \cancel{16} - \cancel{8zw} + 6izw + 6iz - \cancel{8z} - 12iw - 12i$$