10/10/19. Cauchy Integral formula. [Unit-III].

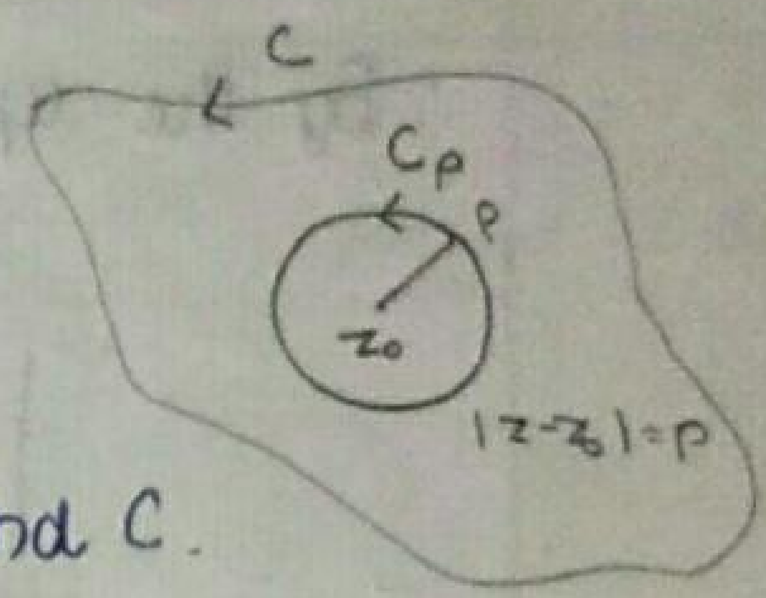
Theorem:

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad — ①$$

Proof:

Let C_ρ be a positively oriented circle $|z-z_0| = \rho$ where ρ is small enough that C_ρ is interior to C as shown in figure.



Since $\frac{f(z)}{z-z_0}$ is analytic between and on the contour C_ρ and C.

By principle of deformation of paths,

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho}\frac{f(z)}{z-z_0}\,dz \quad — ②$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - \int_{C_\rho}\frac{f(z_0)}{z-z_0}\,dz = \int_{C_\rho}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz$$

$— ③$

We know that,

$$\int_{C_\rho}\frac{dz}{z-z_0} = \int_0^{2\pi}\frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta = i[2\pi - 0]$$

$$= 2\pi i \qquad \langle \because z-z_0 = \rho e^{i\theta}\rangle.$$

$$\therefore ③ \Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0)\cdot 2\pi i = \int_{C_\rho}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz.$$

$— ④$

Since f is analytic inside and on C, it is continuous at z_0.

Hence, for each positive number ε, there is a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z-z_0| < \delta. \quad — (5)$$

Choose the radius ρ of the circle C_ρ such that $\rho < \delta$.

When z is on C_ρ, $|z-z_0| = \rho < \delta$.

$\therefore$ (5) $\Rightarrow$ $|f(z) - f(z_0)| < \varepsilon$, for all z on C_ρ.

$\therefore$ By the upper bound for module of contour integrals,

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z-z_0}\, dz \right| \leq \frac{\varepsilon}{\rho} \times 2\pi\rho = 2\pi\varepsilon$$

Hence, (4) $\Rightarrow$

$$\left| \int_C \frac{f(z)}{z-z_0}\, dz - f(z_0)\, 2\pi i \right| < 2\pi\varepsilon \quad — (6)$$

Since, LHS of (6) is a non-negative constant less than an arbitary small positive number, it must be equal to zero.

$$\therefore \int_C \frac{f(z)}{z-z_0}\, dz = f(z_0)\, 2\pi i.$$

(ie) $$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz.$$

Hence, the theorem.

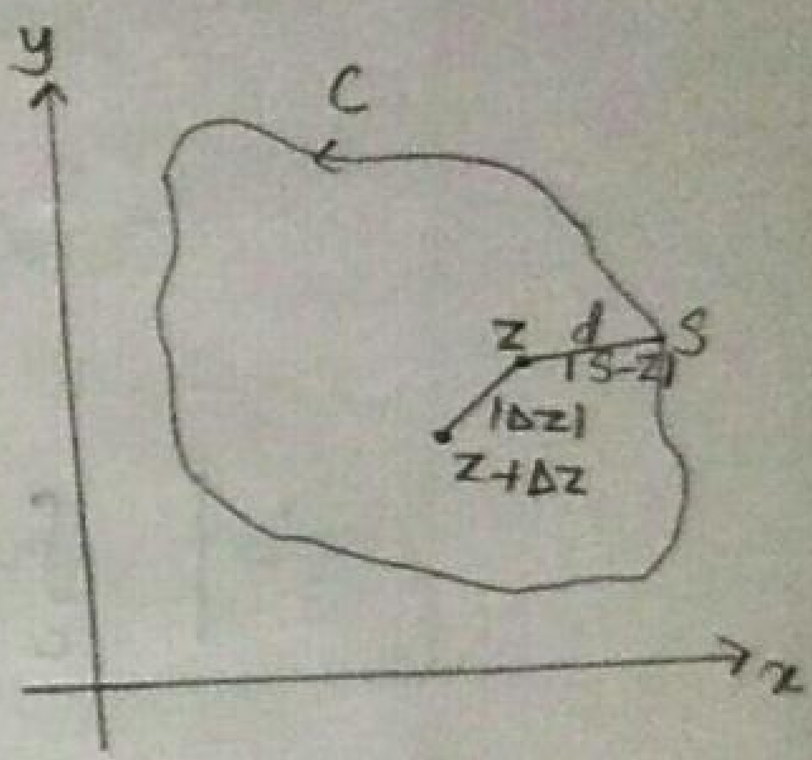
Cauchy integral formula for first derivative

Theorem:

Let f be analytic everywhere inside and on a simple closed contour C taken in the positive sense. then

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds.$$

where z is interior to C and s denotes points on C.



proof:

Let d denote the smallest distance from z to points s on C.

Using Cauchy's integral formula for the points z and $z+\Delta z$, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds \quad \text{--- (1)}$$

and $$f(z+\Delta z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z-\Delta z)}\, ds \quad \text{--- (2)}$$

$$\therefore \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right) \frac{f(s)}{\Delta z}\, ds$$

$$= \frac{1}{2\pi i} \int_C \left(\frac{\Delta z}{(s-z-\Delta z)(s-z)} \right) \frac{f(s)}{\Delta z}\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z-\Delta z)(s-z)}\, ds \quad \text{--- (3)}$$

where $0 < |\Delta z| < d$,

$$\Rightarrow \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds.$$

$$= \frac{1}{2\pi i} \int_C \left(\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} \right) f(s)\, ds$$

$$\left[\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds\right] = \frac{1}{2\pi i}\int_C \frac{\Delta z}{(s-z-\Delta z)(s-z)^2} f(s)\,ds \quad \text{--- (4)}$$

Let M denote the maximum value of $|f(s)|$ on C.

since $|s-z| \geq d$ and $|\Delta z| < d$,

$$|s-z-\Delta z| \geq \big||s-z| - |\Delta z|\big|$$

$$\geq |d - |\Delta z||$$

$$|s-z-\Delta z| > 0.$$

$$\therefore \left|\int_C \frac{(\Delta z)\, f(s)}{(s-z-\Delta z)(s-z)^2}\,ds\right| \leq \frac{|\Delta z|\cdot M}{d^2(d-|\Delta z|)} L \quad \text{--- (5)}$$

where L is the length of C.

Taking $\Delta z \to 0$,

$\therefore$ The RHS of (5) tends to 0

where (4) $\Rightarrow$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds = 0$$

$$\therefore f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds.$$ Hence proved.

Note: Cauchy integral formula for n^{th} derivative.

Let f be analytic everywhere inside and on a simple closed contour C taken in the positive sense. then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds \quad (n = 0, 1, \dots).$$

where z is interior to C and s denotes point on C.

proof:

For $n=0$, $f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds$

For $n=1$, $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds.$

Hence by induction on n, we can prove that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds.$$

Example: 1

If c is the positively oriented unit circle $|z|=1$ then prove that $\int_C \frac{e^{2z}}{z^4}\, dz = \frac{8\pi i}{3}.$

Solu,

$f(z) = e^{2z}.$

$\therefore f'(z) = 2e^{2z}$; $f''(z) = 4e^{2z}$; $f'''(z) = 8e^{2z}.$

By Cauchy integral formula,

$$f'''(z_0) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{3+1}}\, dz$$

$$\Rightarrow f'''(0) = \frac{\cancel{6}\,3}{\cancel{2}\pi i} \int_C \frac{e^{2z}}{z^4}\, dz.$$

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$$8 = \frac{3}{\pi i} \int_c \frac{e^{2z}}{z^4} dz$$

$$\therefore \int_c \frac{e^{2z}}{z^4} dz = \frac{8\pi i}{3}.$$

Example: 2.

Let z_0 be any point interior to a positive oriented simple closed contour C. when $f(z) = 1$,

$$\int_c \frac{dz}{z - z_0} = 2\pi i \quad \text{and}$$

$$\int_c \frac{dz}{(z - z_0)^{n+1}} = 0 \quad (n = 1, 2, \cdots)$$

Proof:

By Cauchy- integral formula,

$$\int_c \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f(z_0)$$

where $f(z) = 1$, $f(z_0) = 1$.

$$\therefore \int_c \frac{1}{z - z_0} dz = 2\pi i.$$

By Cauchy-integral formula for n^{th} derivatives,

$$\int_c \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} \cdot f^{(n)}(z_0)$$

where $f(z) = 1$; $f(z_0) = 1 \Rightarrow f^{(n)}(z_0) = 0 \ (n = 1, 2, \cdots)$

$$\therefore \int_c \frac{1}{(z - z_0)^{n+1}} dz = 0 \quad (n = 1, 2, \cdots).$$

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11/10/19. Some Consequence of the extension

Theorem:

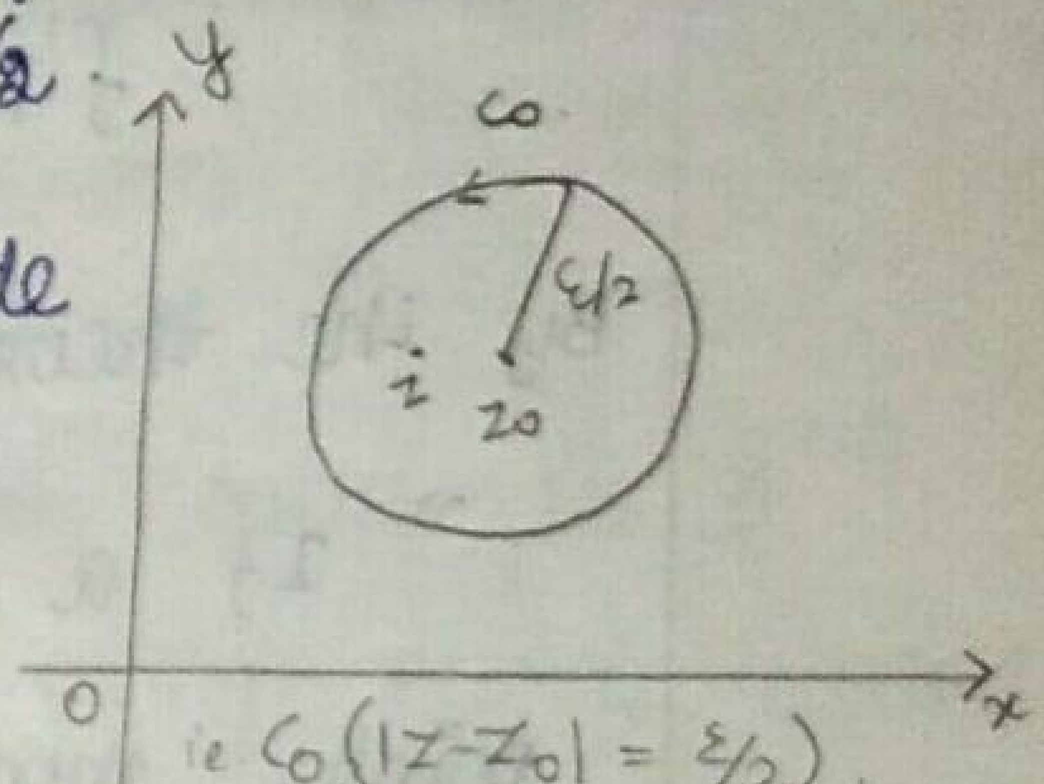
If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Proof:

Assume that, a function f is analytic at a point z_0.

Then, there must be a neighborhood $|z-z_0| < \varepsilon$ of z_0, throughout which f is analytic.

Let C_0 be the positively oriented circle center at z_0 with radius $\varepsilon/2$.



ie $C_0 (|z-z_0| = \varepsilon/2)$.

Then, f is analytic inside and on C_0.

$\therefore$ By the Cauchy's - Integral formula for second derivative,

$$f''(z) = \frac{2!}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)^3} ds$$

$$\therefore f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)}{(s-z)^3} ds$$, for all z interior to C_0

Hence, $f''(z)$ exists for all z in the neighbourhood $|z-z_0| < \varepsilon$.

$\therefore f'$ is analytic at z_0. If we apply the Same argument to the analytic function f', we can prove f'' is analytic at z_0 and so on.

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Hence, the derivatives of all orders of function f are analytic at z_0.

Thus, proved the theorem.

Corollary:

If a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic at a point $z = (x,y)$ then the component functions u and v have continuous partial derivatives of all orders at that point.

Proof:

Since f is analytic, it is continuous and

$$f'(z) = u_x + i v_x = v_y - i u_y .$$

By the theorem,

" If a function f is analytic at a given point, then its derivatives of all orders are analytic there too".

we get,

the derivative of all orders of f are analytic.

Since $f'(z)$ is analytic, it is continuous.

$\therefore$ The first order partial derivatives of u and v are continuous.

Since $f''(z)$ is analytic, it is continuous and $f''(z) = u_{xx} + i v_{xx} = v_{yx} - i u_{yx}$.

$\therefore$ The second order partial derivatives of u and v are continuous.

In general,

The components functions u and v have continuous partial derivative of all orders.

Hence, proved the theorem.

13/10/19 **Morera's theorem:**

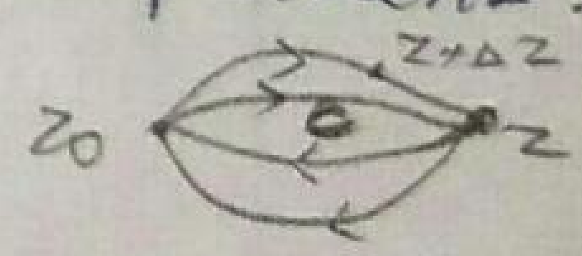
Let f be continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour C in D, then f is analytic throughout D.

proof:

Let z_0 be a fixed point in D.

Since the integral along every closed contour is zero, the integral

$$\int_{z_0}^{z} f(z)\,dz$$ is path independent.



Let $F(z) = \int_{z_0}^{z} \overset{f(s)\,ds}{f(z)\,dz}$ — ①

Clearly, F(z) is a single valued function in D.

consider, $\dfrac{F(z+\Delta z) - F(z)}{\Delta z} = \dfrac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds\right]$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_z^{z+\Delta z} f(s)\, ds\right]$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} \left(f(s) - f(z) + f(z)\right) ds$$

where $\displaystyle\int_z^{z+\Delta z} \frac{f(z)}{\Delta z}\, ds = \frac{f(z)}{\Delta z} \times [s]_z^{z+\Delta z} = \frac{f(z)}{\Delta z} \times (z+\Delta z - z) = \frac{f(z)}{\Delta z} \times \Delta z = f(z).$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \int_z^{z+\Delta z} \left(\frac{f(s) - f(z)}{\Delta z}\right) ds \quad \text{—(2)}$$

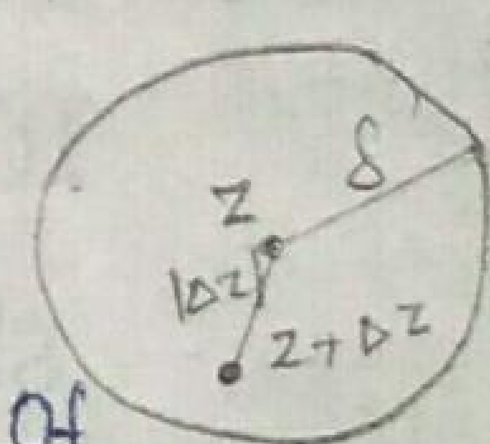
Since f is continuous in D. $\lim_{s \to z} f(s) = f(z)$

$\therefore$ Given $\varepsilon > 0$, we can find a $\delta > 0$

$\ni |f(s) - f(z)| < \varepsilon$ whenever $|s - z| < \delta$ —(3)

If we choose $|\Delta z| < \delta$, then $z + \Delta z$

lie in $|s - z| < \delta$



$\therefore$ By upper bound for moduli of contour integrals,

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \left|\int_z^{z+\Delta z} \left(\frac{f(s) - f(z)}{\Delta z}\right) ds\right|$$

$$\leq \frac{\varepsilon}{|\Delta z|} \cdot |\Delta z|$$

$$\therefore \quad \left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \leq \varepsilon.$$

Taking $\lim_{\Delta z \to 0}$, we get

$$F'(z) = f(z).$$

(ie) $F'(z)$ exist through out D. Hence $F(z)$ is analytic.

Then by the theorem

" If a function f is analytic at a given point, then the derivative of all orders are analytic there too".

$\therefore$ we have $F'(z)$ is analytic.

Hence, $f(z)$ is analytic throughout D.

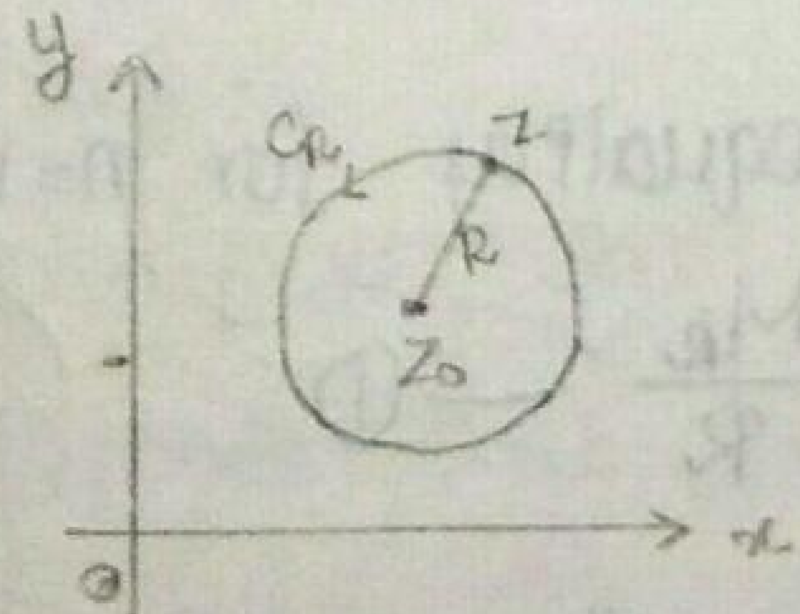
Thus proved.

13/10/19.

Cauchy's Inequality Theorem:

Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R as in figure. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\, M_R}{R^n} \quad (n=1,2,\ldots).$$



Proof:

By extension of Cauchy's Integral formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n=1,2,\ldots)$$

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By the upper bound for moduli of contour integral

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\right| \cdot \left| \int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \right| \quad (|i|=1)$$

$$\leq \frac{n!}{2\pi} \times \frac{M_R}{R^{n+1}} \cdot 2\pi R$$

$$\leq \frac{n!\, M_R}{R^n} \quad (n=1,2,\ldots)$$

(ie) $$|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n} \quad (n=1,2,\ldots)$$

Hence, proved the theorem.

Liouville's theorem.

If a function f is entire and bounded in the complex plane, then $f(z)$ is constant througout the plane.

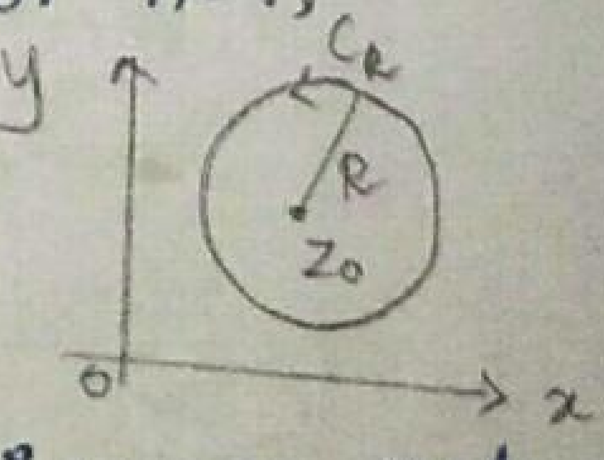
Proof:

Since f is entire, we can apply Cauchys inequality for any choice of z_0 and R.

Let $C_R : |z-z_0| = R$.

Using Cauchys inequality for $n=1$,

we have $|f'(z_0)| \leq \frac{M_R}{R}$ — ①



where M_R denotes the maximum value of $|f(z_0)|$ on C_R.

Since f is bounded, there is a nonnegative constant M such that $|f(z)| \leq M$ for all z.

$\therefore |f(z)| \leq M$ for all z on C_R (ie) $M_R \leq M$

This implies $|f'(z_0)| \leq \frac{M}{R}$ — ②

where R can be arbitrarily large in the inequality ②, M is independent of R.

Hence, ② holds for arbitrarily large value of R only if $f'(z_0) = 0$

Since z_0 was arbitrary, $f'(z) = 0$ for all z in the complex plane.

Hence, $f(z)$ is constant throughout the complex plane.

Fundamental theorem of algebra:

Any polynomial, $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ ($a_n \neq 0$) of degree n ($n \geq 1$) has atleast one zero. That is, there exists atleast one point z_0 such that $P(z_0) = 0$.

Proof:

We shall prove this theorem by contradiction. Suppose that $P(z) \neq 0$ for any value of z.

Let $f(z) = \frac{1}{P(z)}$ — ①

Since $P(z)$ is entire and non-zero for any value of z, $f(z)$ is entire.

We shall prove that $f(z)$ is bounded in the entire complex plane.

$$\text{Let } \omega = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \quad \text{—} \ ②$$

$$\therefore \frac{P(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n$$

$$= \omega + a_n$$

$$P(z) = (\omega + a_n) z^n \quad \text{—} \ ③$$

$\lim\limits_{z\to\infty} \omega = 0 \Rightarrow \lim\limits_{z\to\infty} \left[\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right] = 0 \Rightarrow \lim\limits_{z\to\infty} \frac{a_0}{z^n} = 0$

$\varepsilon = \frac{|a_n|}{2n}, \exists\ \delta_1 > 0 \ni \left|\frac{a_0}{z^n}\right| < \frac{|a_n|}{2n} \ (|z| > 1/\delta_1 = R_1)$ / $\lim\limits_{z\to\infty} \frac{a_1}{z^{n-1}} = 0$; $\varepsilon = \frac{|a_n|}{2n}\ \exists\ \delta_2 > 0 \ni \left|\frac{a_1}{z^{n-1}}\right| < \frac{|a_n|}{2n}$ $(|z| > 1/\delta_2 = R_2)$ / $\lim\limits_{z\to\infty} \frac{a_{n-1}}{z} = 0$; $\varepsilon = \frac{|a_n|}{2n}\ \exists\ \delta_n > 0 \ni \left|\frac{a_{n-1}}{z}\right| < \frac{|a_n|}{2n}$ $(|z| > \frac{1}{\delta_n} = R_n)$

Let $R = \max(R_1, R_2, \ldots R_n)$

$\left|\frac{a_0}{z^n}\right| < \frac{|a_n|}{2n}; \ldots \left|\frac{a_{n-1}}{z}\right| < \frac{|a_n|}{2n}$

← [Since each term in ω tends to zero as $z \to \infty$, we can find a sufficiently large positive R] such that

$$\left|\frac{a_0}{z^n}\right| < \frac{|a_n|}{2n}, \left|\frac{a_1}{z^{n-1}}\right| < \frac{|a_n|}{2n}, \ldots$$

$$\ldots \left|\frac{a_{n-1}}{z}\right| < \frac{|a_n|}{2n} \quad \text{whenever } |z| > R.$$

$$|\omega| \leq \left|\frac{a_0}{z^n}\right| + \left|\frac{a_1}{z^{n-1}}\right| + \cdots + \left|\frac{a_{n-1}}{z}\right|$$

$$< \cancel{n} \cdot \frac{|a_n|}{2\cancel{n}} \quad \text{whenever } |z| > R.$$

$$\therefore |\omega| < \frac{|a_n|}{2} \quad \text{whenever } |z| > R$$

$$\boxed{\frac{1}{|\omega|} > \frac{2}{|a_n|} \Big/ -|\omega| > -\frac{|a_n|}{2}}$$

Consequently,

$$\begin{pmatrix} |a-b| \geq \big||a|-|b|\big| \\ |a+b| \geq \big||a|-|b|\big| \end{pmatrix}$$

$$|a_n + \omega| \geq \big||a_n| - |\omega|\big|$$

$$\geq |a_n| - |\omega|$$

$$> |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2}.$$

$\therefore |a_n + w| > \frac{|a_n|}{2}$ whenever $|z| > R$.

(3) $\Rightarrow$ $|p(z)| = |a_n + w| \cdot |z^n|$

$$> \frac{|a_n|}{2} \cdot |z|^n$$

$$|p(z)| > \frac{|a_n|}{2} \cdot R^n \quad \text{whenever } |z| > R$$

Hence, $|f(z)| = \frac{1}{|p(z)|} < \frac{2}{|a_n| \cdot R^n}$ whenever $|z| > R$.

$\therefore f(z)$ is bounded in $|z| > R$.

But, $f(z)$ is continuous in closed bounded disk $|z| \le R$.

By the theorem,

" If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that

$|f(z)| \le M$ for all points z in R,

where equality holds for atleast one such z."

we get, $f(z)$ is bounded in the disk $|z| \le R$.

Hence, $f(z)$ is bounded in the entire comple plane.

Then By the liouville's theorem,

$f(z) = \frac{1}{p(z)}$ is constant.

This implies $p(z)$ is constant. This contradicts to the fact that $p(z)$ is not constant.

Hence, there exist atleast one z_0. Such that $p(z_0) = 0$. Hence the theorem.

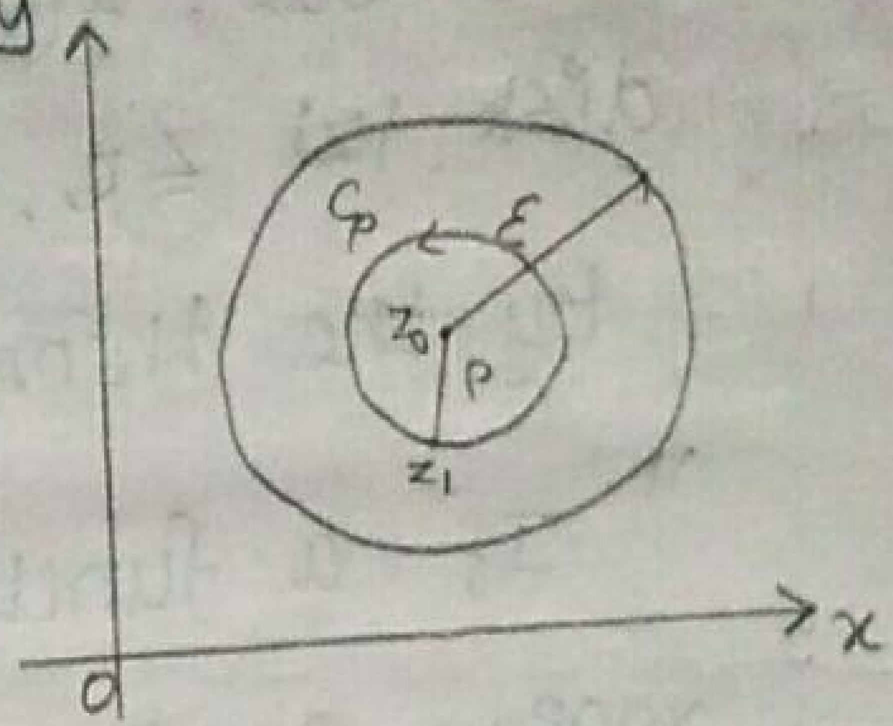
16/10/19

Maximum modulus Principle:

Lemma: Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighbourhood $|z-z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Proof:

Assume the hypothesis that $|f(z)| \leq |f(z_0)|$ for every z in $|z-z_0| < \varepsilon$ and f is analytic in $|z-z_0| < \varepsilon$.



Let z_1 be any point other than z_0 in the neighbourhood $|z-z_0| < \varepsilon$. Let ρ be the distance between z_0 and z_1 and let C_ρ denotes the positively oriented circle $|z-z_0| = \rho$, centered at z_0 and passing through z_1.

Since f is analytic in $|z-z_0| < \varepsilon$, f is analytic in C_ρ.

$\therefore$ using Cauchy's Integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)\,dz}{z-z_0} \quad \text{—} \quad ①$$

The parametric repersentation of C_ρ is

$$z = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

Then ① $\Rightarrow$ $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$ — ②

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta \quad \text{--- (3)}$$

on the other hand, $|f(z)| \leq |f(z_0)|$

$$|f(z_0+\rho e^{i\theta})| \leq |f(z_0)| \quad , 0 \leq \theta \leq 2\pi \quad \text{--- (4)}$$

$\therefore$ (4) $\Rightarrow$

$$\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta \leq \int_0^{2\pi} |f(z_0)|\, d\theta$$

$$\leq 2\pi\, |f(z_0)|$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta \leq |f(z_0)|$$

(ie) $$|f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta \quad \text{--- (5)}$$

From (3) and (5),

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$

$$2\pi\, |f(z_0)| = \int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$

$$\int_0^{2\pi} |f(z_0)|\, d\theta = \int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta .$$

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0+\rho e^{i\theta})| \right] d\theta = 0 \quad \text{--- (6)}$$

Since $|f(z_0)| - |f(z_0+\rho e^{i\theta})|$ is continuous in the variable θ and $|f(z_0)| - |f(z_0+\rho e^{i\theta})| \geq 0$ in $0 \leq \theta \leq 2\pi$.

In the view of (6),

$$|f(z_0)| - |f(z_0+\rho e^{i\theta})| = 0$$

(ie) $$|f(z_0+\rho e^{i\theta})| = |f(z_0)| \quad , 0 \leq \theta \leq 2\pi .$$

This shows that $|f(z)| = |f(z_0)|$ for all points z on the circle $|z-z_0| = \rho$.

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Since z_1 is any point in the deleted neighborhood $0 < |z-z_0| < \varepsilon$,

$|f(z)| = |f(z_0)|$ is satisfied by all points z lying on any circle $|z-z_0| = \rho$, where $0 < \rho < \varepsilon$.

Consequently, $|f(z)| = |f(z_0)|$ everywhere in the neighbourhood $|z-z_0| < \varepsilon$.

(ie) $|f(z)|$ is constant throughout $|z-z_0|<\varepsilon$.

We know that, when the modulus of an analytic function is constant in a domain, then function itself is constant there.

Since $f(z)$ is analytic in $|z-z_0| < \varepsilon$,
$\therefore$ $f(z)$ is constant throughout the neighborhood $|z-z_0| < \varepsilon$.

Hence, $f(z) = f(z_0)$ for all z in the neighbor-hood $|z-z_0| < \varepsilon$.

Hence, the proof.

Maximum modulus Principle.

Theorem:

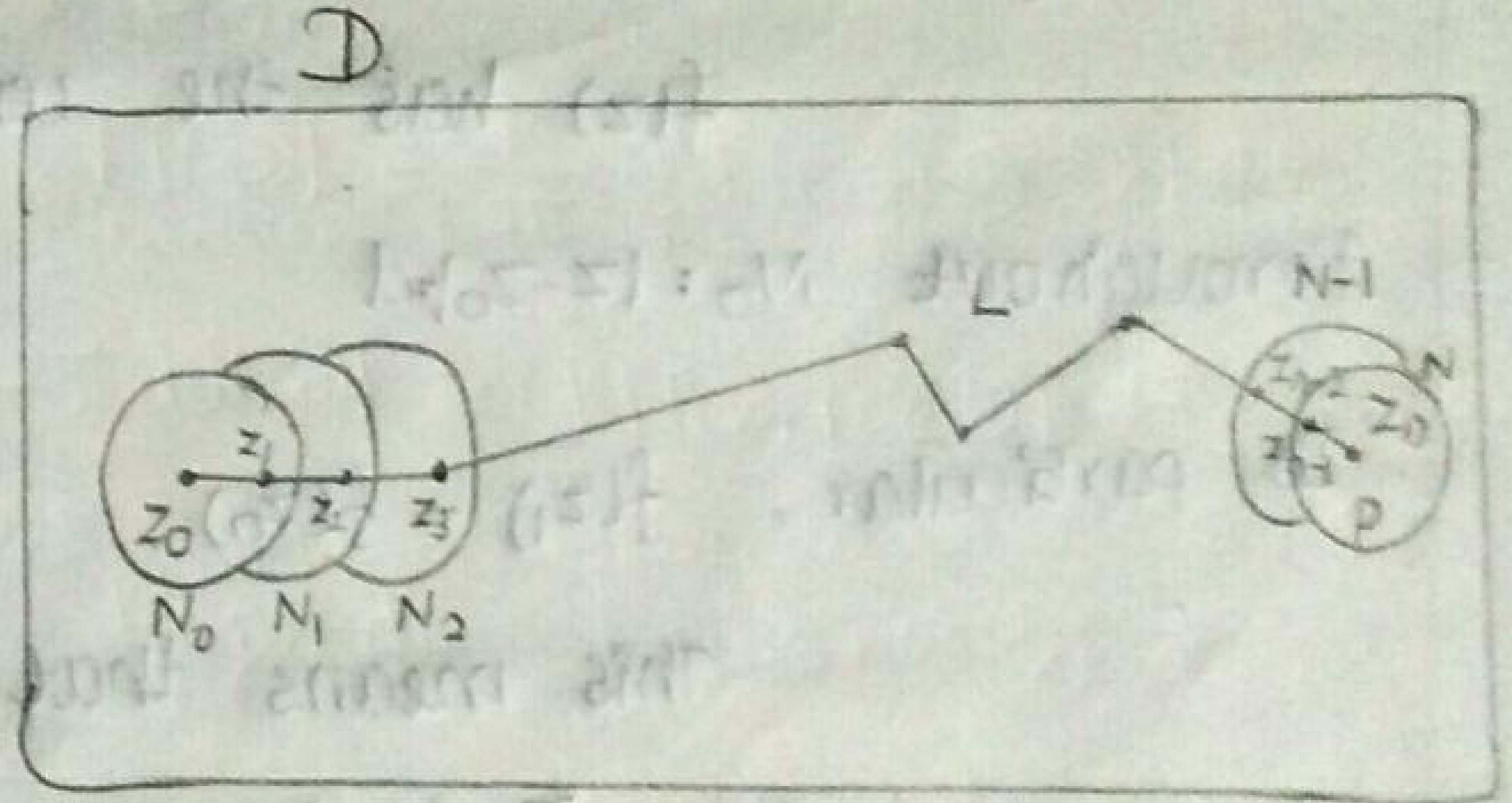
If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That in, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

Proof:

Given that, f is analytic in D.

we prove the theorem by assuming that $|f(z)|$ has a maximum value at some point z_0 in D and we shall prove that $f(z)$ must be constant throughout D.

Since D is a connected open set, we can Draw a polygonal line L lying in D from z_0 to any other point P in D as shown in figure.



Let d denotes the shortest distance from points on L to the boundary of D.

When D is the entire plane, d may have any positive value.

Hence, there is a finite sequence of points $z_0, z_1, z_2, z_3, \ldots\ldots, z_{n-1}, z_n$ along L. Such that z_n concicides with the point P and

$$|z_k - z_{k-1}| < d \quad (k=1,2,\ldots).$$

Finally, we formed a finite sequence of neighbothoods $N_0, N_1, N_2, \ldots\ldots\ldots, N_n$, where each N_k has center z_k and radius d.

Since all the neighborhoods ~~and~~ are contained in D, $f(z)$ is analytic in each of these neighborhoods.

Also the center of each neighborhood N_k $(k=1,2,\cdots n)$ lies in the N_{k-1} neighborhood.

By the assumption, $|f(z)| \leq |f(z_0)|$ for all z in D,

$|f(z)| \leq |f(z_0)|$ for all z in $N_0 : |z-z_0|=d$

Hence by the lemma,

$f(z)$ has the constant value $f(z_0)$ throughout $N_0 : |z-z_0|=d$

In particular, $f(z_1) = f(z_0)$.

This means that $|f(z)| \leq |f(z_1)|$ for each point z_1 in N_1.

again by the same lemma,

$f(z) = f(z_1) = f(z_0)$ for all point z in N_1. Since z_2 is in N_1, $f(z_2) = f(z_0)$.

Hence $|f(z)| \leq |f(z_2)|$ for each point z in N_2.

Once again by the lemma,

$f(z) = f(z_2) = f(z_0)$ for all point z in N_2.

Continuing in this manner, we eventually reach the neighborhood N_n and arrive at the fact,

$$f(z) = f(z_n) = f(z_0).$$

Since z_n coincide with P, which is any point other than z_0 in D.

We conclude that, $f(z) = f(z_0)$ for every point z in D.

Hence $f(z)$ is constant throughout D.

Hence, the theorem.