Chapter 23

Second Order Differential Equations with Constant Coefficients

A second order linear differential equation is an equation that can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x) \qquad ...(1)$$

Here $a_2(x), a_1(x), a_0(x)$ and $b(x)$ are continuous functions of x on an intevnal D. When a_0, a_1, a_2 are constants, we say that the equation has constant co-efficients; otherwise it has variable co-efficients.

Now we are interested in the linear equation where $a_2(x) \neq 0$. In that case, we can write the above equation in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = g(x) \qquad ...(2)$$

where $p(x) = \dfrac{a_1(x)}{a_2(x)}$ and $q(x) = \dfrac{a_0(x)}{a_2(x)}$

Associated with each equation (2) is the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \qquad ...(3)$$

which is obtained from (2) by replacing $g(x)$ by zero. We say that (2) is a non-homogeneous equation and that (3) is the corresponding homogeneous equation. Here the meaning 'homogeneous' is not related to the use of the term for the first order equation but is used in the sense of homogeneous system of linear equations as studied in linear algebra.

A linear differential equation of n^{th} order is of the form

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_1\frac{dy}{dx} + a_0 = F(x) \qquad ...(4)$$

where $a_n, a_{n-1}, \ldots\ldots a_1, a_0$ depend only on x or constants.

If $a_n, a_{n-1} \ldots\ldots a_1, a_0$ are constants, then equation (4) is called a linear differential equation with constant co-efficients. When $n = 2$,

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0 \qquad ...(5)$$

is called a linear differential equation of order two. If we use the differential operator D for $\frac{d}{dx}$ so that $Dy = \frac{dy}{dx}$ then equation (4) can be written as

$$a_n D^n y + a_{n-1} D^{n-1} y + a_{n-2} D^{n-2} y + \ldots\ldots + a_1 Dy + a_0 y = F(x) \qquad ...(6)$$

This equation can also be expressed in the form

$$(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots\ldots + a_1 D + a_0)\,y = F(x) \qquad ...(7)$$

Here it is taken that each term in the parentheses operates on y.

In brief, the above equation can be written as

$$\phi(D)(y) = F(x)$$

where $\phi(D) = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_1 D + a_0 \qquad ...(8)$

For example $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$ is written in the form

$\phi(D)\,y = F(x)$

where $\phi(D) = D^2 - 3D + 2$ and $F(x) = \sin x$

Linear operator

Any operator D^n is called a linear operator

$$\left.\begin{aligned} \text{if } D^n(u+v) &= D^n(u) + D^n v \\ \text{and } D^n(cu) &= c\,D^n(u) \end{aligned}\right\} \qquad ...(9)$$

where u and v are differentiable functions of x and c is a constant. Also n is a positive integer. It can be shown that $\phi(D)$ is also a linear operator. It satisfies the conditions

$$\left.\begin{aligned} \phi(D)(u+v) &= \phi(D)\,u + \phi(D)\,v \\ \text{and } \phi(D)(cu) &= c\,\phi(D)\,u \end{aligned}\right\} \qquad ...(10)$$

Our interest is to find the solution of the equation

$$\phi(D)\,y = F(x) \qquad ...(11)$$

The associated homogeneous equation is

$$\phi(D)\,y = 0 \qquad ...(12)$$

We shall first solve the equation (12) and then by its help, we obtain the solution of equation (11).

The differential operator D or D^n behaves like an algebraic quantity as given below:

$$D^m (D^n) y = D^n (D^m) y \quad \text{(Index law)}$$

$$(D - m_1)(D - m_2) y = (D - m_2)(D - m_1) y \quad \text{(commutative law)}$$

$$(D - m_1)(D - m_2) y = [D^2 - (m_1 + m_2) D + m_1 m_2] y \quad \text{(Factor law)}$$

Factor law further emphazises that any polynomial in operator D i e $\phi(D)$ can be factored if the co-efficients in the polynomial are constants.

$$i\,e\ \phi(D) = D^2 - 7D + 6 = (D - 1)(D - 6)$$

If $\quad \phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots$

$\ldots\ldots + a_{n-1} D + a_n$, then

we can factorise $\phi(D)$ as

$$\phi(D) = a_0 (D - m_1)(D - m_2) \ldots\ldots (D - m_n)$$

Solution of $\phi(D) y = 0$.

If $\phi(D)$ is a polynomial of degree n in D, factoring $\phi(D)$ in terms of linear factors we have

$$\phi(D) = a_0 (D - m_1)(D - m_2) \ldots (D - m_n) = 0$$

The roots of $\phi(D) = 0$ are $m_1, m_2 .. m_n$ then the homogeneous equation (auxiliary equation) in terms of these functions is

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n) y = 0$$

This equation will be satisfied by any one of them say

$$(D - m_1) y = 0.$$

$$(D - m_1) y = 0 \Rightarrow \frac{dy}{dx} - m_1 y = 0$$

$$\frac{dy}{y} = m_1\, dx \text{ or } y_1 = c_1 e^{m_1 x}$$

Similarly $y = c_2 e^{m_2 x}$, $y = c_3 e^{m_3 x} \ldots\ldots y_n = C_n e^{m_n x}$ also individually satisfy the equation $\phi(D) y = 0$ which are solutions of $(D - m_2) y = 0$, $(D - m_3) y = 0$ etc,

Note : $m_1, m_2, \ldots\ldots m_n$ are the roots of the equation $f(m) = 0$

$f(m) = 0$ is called auxiliary equation.

Theorem :

If y_1 and y_2 are solutions of $\phi(D) = 0$ then $c_1 y_1 + c_2 y_2$ is also a solution.

Proof : Since y_1 and y_2 are solutions of $\phi(D) y = 0$

$$\phi(D) y_1 = 0 \quad \ldots(1)$$

$$\phi(D) y_2 = 0 \quad \ldots(2)$$

Multiplying (1) by c_1 and (2) by c_2 we get

$$\phi(D)(c_1 y_1) = 0$$

$\phi(D)(c_2 y_2) = 0$ since $\phi(D)$ is a linear operator.

Also $\phi(D)(c_1 y_1 + c_2 y_2) = 0$

$\therefore y = c_1 y_1 + c_2 y_2$ is also a solution of $\phi(D) y = 0$

This result can be extended to n solutions $y_1, y_2 \ldots y_n$

i.e If $y_1, y_2 \ldots y_n$ are the solutions of $\phi(D) y = 0$ then

$y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is also solution of $\phi(D) y = 0$

Note : This equation has n arbitrary constants and hence the equation $(D - m_1) \ldots\ldots (D - m_n) y = 0$ has solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}$$

Example 1

Consider the equation

Solution :

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

This equation is in the form

$$(D^2 - 3D + 2) y = 0$$
$$(D - 1)(D - 2) = 0$$
$$D = 1, 2 \text{ or } m_1 = 1, m_2 = 2$$

$\therefore$ The general solution is $y = Ae^x + Be^{2x}$

In the above analysis of the solution, we have taken the roots $m_1, m_2, \ldots\ldots m_n$ as distinct. Let us now discuss the solutions depending upon the nature of the roots of the second order linear differential equation.

Case 1: Consider the linear differential equation

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$(a D^2 + a_1 D + a_0) y = 0$$

If the roots of the equation $a_2 D^2 + a_1 D + a_0 = 0$ are real and distin[ct]

say m_1 and m_1

then the solution of the $a_2 (D - m_1)(D - m_2) y = 0$ is

$$\boxed{y = Ae^{m_1 x} + Be^{m_2 x}}$$

Case 2 : Suppose the roots are real and equal say m, m.

Then the equation is $a_2 (D - m)(D - m) y = 0$. If we write down th[e] solution as

$$y = Ae^{mx} + Be^{mx}$$

$$i\,e\ y = (A + B) e^{mx} = c\, e^{mx}$$

this solution is not the general solution since the number of arbitra[ry] constants is not equal to 2. So to determine the general solution, we wr[ite]

$$(D - m)(D - m) y = 0 \text{ since } a_2 \neq 0$$

$$(D - m) z = 0 \text{ taking } (D - m) y = z$$

$$i.e\ \frac{dz}{dx} - mz = 0$$

The solution is $z = c_1 e^{mx}$

Hence putting the value of z in $(D - m)y = z$, we have

$$(D - m) y = c_1 e^{mx}$$

$$\frac{dy}{dx} - my = c_1 e^{mx}$$

$$\text{or } y e^{-mx} = \int c_1 e^{mx} e^{-mx} dx + c_2$$

$$= c_1 x + c_2$$

$$y = (c_1 x + c_2) e^{mx}$$

Hence we should take $y = (c_1 x + c_2) e^{mx}$ as the general solution of the differential equation.

Case 3: Suppose the roots are imaginery say $\alpha \pm i\beta$

$$i\ e\ m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= c_1 e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$+ c_2 e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$= e^{\alpha} x .(A \cos \beta x + \beta \sin \beta x)$$

Solution for non-homogeneous equation

If $y = u(x)$ is the solution of $\phi(D) y = F(x)$ and $y = v(x)$ is the solution of $\phi(D) y = 0$ then the complete solution of $\phi(D) y = F(x)$ is $y = u(x) + v(x)$

Proof : since $y = u(x)$ is solution of $\phi(D) y = F(x)$

we have $\phi(D) u(x) = F(x)$ ...(1)

$y = v(x)$ is the solution of $\phi(D) y = 0$

$$\phi(D) v(x) = 0 \quad ...(2)$$

Adding (1) and (2),

$$\phi(D) u(x) + \phi(D) v(x) = F(x)$$

$$\phi(D) [u + v] = F(x)$$

$\therefore y = u(x) + v(x)$ is a solution of $\phi(D) y = F(x)$

The solution $y = v(x)$ will contain n arbitrary constants and is called the complementary solution.

If $y = u(x)$ has no arbitray constant, it is called a particular solution of $\phi(D) y = F(x)$.

Then $y = u(x) + v(x)$ will also contain n arbitrary constants and is called the complete solution or general solution of $\phi(D)(y) = F(x)$. $u(x)$ is called complementary function of the given DE and $v(x)$ is called the particular integral of the given DE.

Hence for the homogeneous equation,

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_o y = 0$$

$$i\ e\ a_2 D^2 + a_1 D + a_o y = 0$$

i) If the auxiliary equation has real and distinct roots m_1, m_2, the olution is $y = Ae^{m_1 x} + B e^{m_2 x}$

ii) If the roots are equal say m and m, then the solution is $y = e^{mx}(Ax + B)$.

iii) If the roots are imaginary say $\alpha \pm i\beta$ then the solution is

iii) $y = e^{\alpha x}(A\cos\beta x + \beta \sin\beta x)$

Example 1

Solve $(D^2 - 4D + 3)y = 0$

Solution :

The auxiliary equation is

$$m^2 - 4m + 3 = 0$$

$$i.e. (m-1)(m-3)y = 0$$

∴ the roots are 1, 3,

∴ the general solution is $y = Ae^x + Be^{3x}$

Example 2 :

Solve $(D^2 - 6D + 9)y = 0$

Solution :

The auxiliary equation is $(D-3)^2 y = 0$

∴the roots are $m = 3, 3$

∴ the general solution is $y = e^{3x}(Ax + B)$

Example 3

Solve $(D^2 + D + 1)y + 0$

Solution :

the auxiliary equation is $m^2 + m + 1 = 0$

the roots are $m = \dfrac{-1 \pm i\sqrt{3}}{2}$

∴ the solution is $y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

Methods of obtaining the particular integral:

For any non-homogeneous DE the general solution is $y = y_c + y_p$ where y_c is the complementary function and y_p is the particular integral.

Now we look into the following methods of finding particular integral

1. Short cut methods involving operators

2. Method of variation of parameters.

First we study the method involving operators

I. Inverse operators : $\dfrac{1}{D}, \dfrac{1}{D-\alpha}$ and $\dfrac{1}{\phi(D)}$.

Consider the equation $Dy = x$ ...(1)

From this, we symbolically write $y = \dfrac{1}{D}x$ ...(2)

the question that arises is what the equation means :

If we integrate (1) we get

$$y = \int x\,dx = \frac{x^2}{2} + c.$$

from this we learn the definition that

$$\frac{1}{D}x = \int x\,dx$$

We do not attach the constant of integration since it is the PI.

Thus the operation of multiplying by $\dfrac{1}{D}$ means integration of the function. Similarly the meaning of $\dfrac{1}{D^2}$ is integrating the function twice and $\dfrac{1}{D^3}$ means integrate the function twice and so on.

Consider the operator $\dfrac{1}{D}$

Consider the equation $(D-\alpha)y = f(x)$ where α is a constant.

Let us formally write $y = \dfrac{1}{D-\alpha}f(x)$

Since (1) is a linear DE, its solution is

$$y = e^{\alpha x}\int e^{-\alpha x} f(x)\,dx \quad ...(2)$$

Thus we define $\dfrac{1}{D-\alpha}f(x) = e^{\alpha x}\int e^{-\alpha x} f(x)\,dx$

Taking $\alpha = 0$, we have

$$\frac{1}{D}f(x) = \int f(x)\,dx$$

Next we consider $(D-\alpha_1)(D-\alpha_2)y = f(x)$ where α_1 and α_2 are constants. Here we symbolically write the above equation as

$$y = \frac{1}{(D-\alpha_1)(D-\alpha_2)} f(x)$$

This means $y = \frac{1}{D-\alpha_1}\left[\frac{1}{D-\alpha_2} f(x)\right]$

$$= \frac{1}{D-\alpha_1}\left[e^{\alpha_2 x}\int e^{-\alpha_2 x} f(x)\,dx\right]$$

$$= e^{\alpha_1 x}\int e^{-\alpha_1 x}\left[e^{\alpha_2 x}\int e^{-\alpha_2 x} f(x)\,dx\right]$$

Similarly we extend the result

$$\frac{1}{(D-\alpha_1)(D-\alpha_2)\,.....\,(D-\alpha_n)} f(x)$$

$$= e^{\alpha_1 x}\int e^{-\alpha_1 x}\,.\,e^{\alpha_2 x}\int e^{-\alpha_2 x}\,.......$$

$$.......\,e^{\alpha_n x}\int e^{-\alpha_n x} f(x)\,dx^n$$

Since D behaves like algebriac symbol obeying all laws of algebra and $\alpha_1, \alpha_2, \alpha_n$ are all constants, the expression

$\frac{1}{(D-\alpha_1)(D-\alpha_2)\,.....\,(D-\alpha_n)}$ can also be resolved into partial fractions.

$$i.e.\ \frac{1}{(D-\alpha_1)(D-\alpha_2)\,.....\,(D-\alpha_n)}$$

$$= \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \,.....\, \frac{A_n}{D-\alpha_n}$$

Here we assume that $\alpha_1, \alpha_2, \ldots\, \alpha_n$ are called all different. For suitably determined constants.

$$\frac{1}{(D-\alpha_1)(D-\alpha_2)\,.....\,(D-\alpha_n)} f(x)$$

$$= \frac{A_1}{D-\alpha_1} f(x) + \frac{A_2}{D-\alpha_2} f(x)\,.....\,+ \frac{A_n}{D-\alpha_n} f(x)$$

$$= A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x} f(x)\,dx + A_2 e^{\alpha_2 x}\int e^{-\alpha_2 x} f(x)\,dx$$

$$+\,.....\,+ A_n e^{\alpha_n x}\int e^{-\alpha_n x} f(x)\,dx \qquad ...II$$

This procedure is easier than the earlier procedure of evaluating the particular integral in I.

Now let us consider the *DE*

$$\phi(D)(y) = F(x)$$

where $\phi(D) = \left(a_n D^n + a_n D^{n-1} + \,.....\, + a_{n-1} D + a_0\right)$

$a_n, a_1, a_2, \ldots a_0$ being constants. For solving the above equation symbolically we have $y = \frac{1}{\phi(D)} F(x)$

where $\frac{1}{\phi(D)}$ represents the operation to be perfomed on $F(x)$

Here $\phi(D)$ can be factorised and using the methods explained (either directly or by partial fractions) we can obtain $\frac{1}{\phi(D)} f(x)$.

Here $\frac{1}{\phi(D)} f(x)$ is called the particular solution or particular integral of the equation $\phi(D) y = f(x)$ since $\frac{1}{\phi(D)} f(x)$ is that function of x when operated by $\phi(D)$ gives $f(x)$

$$i\,e\ \phi(D)\left[\frac{1}{(D)} f(x)\right] = F(x)$$

Here $y = \frac{1}{\phi(D)} f(x)$ is called particular integral. Here we also note that $\frac{1}{\phi(D)}$ is also an operator like $\frac{1}{D}$ or $\frac{1}{D-\alpha}$

Short cut methods for special types of PI:

Let us look into some short cut methods of evaluating particulat integrals where the function $f(x)$ on the RHS is of special type. We will discuss the following types of special functions which appear on the RHS of the linear DE.

i) $F(x) = e^{ax}$

ii) $F(x) = \sin ax$ or $\cos ax$

iii) $F(x) = \sinh ax$ or $\cosh ax$

iv) $F(x) = x^m$

v) $F(x) = e^{ax} v$ where v is any function of x

vi) $F(x) = x^m \sin ax$ or $x^m \cos ax$

vii) $F(x) = x v$ where v is any function of x

Let us now discuss these cases in evaluating particular integrals:

Case 1: $F(x) = e^{ax}$

We know that $D(e^{ax}) = a e^{ax}$

$$D^2(e^{ax}) = a^2 e^{ax}$$
$$\vdots$$
$$D^n(e^{ax}) = a^n e^x$$

Consider the linear differential equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n) y = e^{ax}$$

Now $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)e^{ax}$

$$= (a_0 a^n + a_1 a^{n-1} + \ldots + a_0)e^{ax}$$

This means

$$i\,e \quad \phi(D) e^{ax} = \phi(a) e^{ax}$$

Operating on both the sides by $\frac{1}{\phi(D)}$ we get

$$\frac{1}{\phi(D)}\left[\phi(D) e^x\right] = \frac{1}{\phi(D)}\left[\phi(a) e^{ax}\right]$$

This implies

$e^{ax} = \phi(a)\frac{1}{\phi(D)} e^{ax}$ since $\frac{1}{\phi(D)}$ is a linear operator

Dividing by $\phi(a)$ we get the formula

$$\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \text{ if } \phi(a) \neq 0$$

When $\phi(a) = 0$, this method fails and we consider the following procedure.

If $\phi(a) = 0$ then by factor theorem, $(D - a)$ is a factor of $\phi(D)$

Let $\phi(D) = (D - a)\,\psi(a)$ where $\psi(a) \neq 0$.

Then
$$\frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\Psi(D)} e^{ax}$$
$$= \frac{1}{D-a}\cdot\frac{e^{ax}}{\Psi(a)}$$
$$= \frac{1}{\Psi(a)}\cdot\frac{1}{D-a} e^{ax}$$
$$= \frac{1}{\Psi(a)} e^{ax}\int e^{-ax} e^{ax}\,dx$$
$$= \frac{1}{\Psi(a)} e^{ax}\int dx$$
$$= x\frac{1}{\Psi(a)} e^{ax}$$

$\therefore$ if $(D - a)$ is a factor of $\phi(D)$,

$$\frac{1}{\phi(D)} e^{(ax)} = x\frac{1}{\Psi(a)} e^{ax} \text{ if } \psi(a) \neq 0$$

Note
$$\phi(D) = (D - a)\,\Psi(D)$$
$$\phi'(D) = (D - a)\,\Psi(D) + \Psi'(D)$$
$$\phi'(a) = 0 + \Psi(D)$$
$$\phi'(a) = \Psi(D)$$

$\therefore$ We can also apply the formula when $D - a$ is a factor of $\phi(D)$.

$$\frac{1}{\phi(D)} e^{ax} = x\frac{1}{\phi(a)} e^{ax}$$

Similarly we can show that if $(D - a)^2$ s a factor of $\phi(D)$

$i\,e$ if $\phi(D) = (D - a)^2 h(D)$, then

$$\frac{1}{\phi(D)} e^{ax} = \frac{x^2}{2}\frac{1}{h(a)} e^{ax}$$

or
$$\frac{1}{\phi(D)} e^{ax} = x^2\frac{1}{\phi''(a)} e^{ax}$$

Case 2 : Suppose $F(x) = \sin ax$ or $\cos ax$

we know that $D(\sin ax) = a\cos ax$

$$D^2(\sin ax) = -a^2\cos ax$$
$$D^3(\sin ax) = -a^3\cos ax$$
$$D^4(\sin ax) = a^4\sin ax$$

$i\,e \quad (D^2)^2 \sin ax = (-a^2)^2 \sin ax$

Similarly $(D^2)^3 \sin ax = (-a^2)^3 \sin ax$

$(D^2)^p \sin ax = (-a^2)^p \sin ax$

i.e we may generalise

$$\phi(D^2) \sin ax = \phi(-a^2) \sin ax$$

Operating on both sides $\frac{1}{\phi(D^2)}$ we get

$$\frac{1}{\phi(D^2)}\left[\phi(D^2) \sin ax\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2) \sin ax\right]$$

$$\text{i e } \sin ax = \phi(-a^2)\frac{1}{\phi(D^2)} \sin ax$$

Dividing by $\phi(-a^2)$ we have

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax \text{ if } \phi(-a^2) \neq 0$$

If $\varphi(-a^2) = 0$, the above procedure fails. In that case, we can proceed as follows:

$\phi(-a^2) = 0$ only if $D^2 + a^2$ is a factor of $\phi(D)$

Let $\phi(D) = (D^2 + a^2)\Psi(D)$

$$\therefore \frac{1}{\phi(D)} \sin ax = \frac{1}{\Psi(D)} \cdot \frac{1}{(D^2 + a^2)} \sin ax$$

First we evaluate $\frac{1}{D^2 + a^2} \sin ax$ and then operate on both sides $\frac{1}{\Psi(D)}$.

Now $\frac{1}{D^2 + a^2} \sin ax = IP \text{ of } \frac{1}{D^2 + a^2} e^{iax}$

$(\because e^{iax} = \cos ax + i \sin ax)$

$$= IP \text{ of } \frac{1}{(D - ai)(D + ai)} e^{iax}$$

$$= IP \text{ of } \frac{1}{D - ai} \cdot \frac{1}{2ai} e^{iax}$$

$$= IP \text{ of } \frac{1}{2ai} \cdot \frac{1}{D - ai} e^{iax}$$

$$= IP \text{ of } \frac{1}{2ai} \cdot x e^{iax}$$

$$= IP \text{ of } -\frac{ix}{2a}(\cos ax + i \sin ax)$$

$$= -\frac{x \cos ax}{2a}$$

The same is the procedure for evaluating

$$\frac{1}{\phi(D)} \cdot \cos ax$$

Case 3 : $F(x) = \sinh ax$ or $\cosh ax$

Proceding as in case 2, we can show that

$$\frac{1}{\phi(D^2)} \sinh ax = \frac{1}{\phi(a^2)} \cosh ax$$

$$\frac{1}{\phi(D^2)} \cosh ax = \frac{1}{\phi(a^2)} \cosh ax$$

Case 4 : Suppose $F(x) = x^m$

Here $\frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$

We can expand $[\phi(D)]^{-1}$ in ascending powers of D as for as the term containing D^m and operate on x^m term by term. Here we do not consider the $(m+1)^{th}$ and higher powers of D since $D^{m+1}(x^m)$ and higher order derivatives of x^m vanish.

Here it will be useful to remember the following binomial expansions.

$$(1 - D)^{-1} = (1 + D + D^2 + \ldots \infty)$$

$$(1 + D)^{-1} = (1 - D + D^2 - D^3 + \ldots \infty)$$

$$(1 - D)^{-2} = (1 + 2D + 3D^2 + \ldots \infty)$$

$$(1 + D)^{-2} = (1 - 2D + 3D^2 - \ldots \infty)$$

Case 5:

Suppose $F(x) = e^{ax} v$ where v is some function of x

Here we have

$$D(e^{ax} v) = e^{ax} Dv + ae^{ax} v$$

$$= e^{ax}\left[D + a\right] v$$

Also $D^2(e^{ax} v) = e^{ax} D^2 v + ae^{ax} Dv + ae^{ax} \cdot (D + a) v$

$$= e^{ax} D^2 v + a e^{ax} \cdot Dv + ae^{ax} Dv + a^2 e^{ax} v$$

$$= e^{ax}\left[D^2 v + 2a Dv + a^2 v\right]$$

$$= e^{ax}\left[D^2 + 2aD + a^2\right]v$$
$$= e^{ax}(D + a)^2 v$$

Similarly proceding we get

$$D^3 e^{ax} v = e^{ax}(D + a)^3 v$$

and in general $D^n(e^{ax} v) = e^{ax}(D + a)^n v$ ...(1)

Here $\phi D(e^{ax} v) = e^{ax}\phi(D + a)v$

Let $\phi(D + a)v = v_1$ ...(2)

Then $v = \dfrac{1}{\phi(D + a)} v_1$

Substituting (2) in (1) we get

$$\phi(D)\left[e^{ax}\frac{1}{\phi(D + a)} v_1\right] = e^{ax} v_1$$

Operating on both the sides by $\dfrac{1}{\phi(D)}$ we get

$$e^{ax}\frac{1}{\phi(D + a)} v_1 = \frac{1}{\phi(D)} \cdot e^{ax} v_1$$

and in general

$$\frac{1}{\phi(D)}(e^{ax} v) = e^{ax} \cdot \frac{1}{\phi(D + a)} v$$

Case 6:

Suppose we have to determine the PI

$$\frac{1}{\phi(D)} \cdot x^m \sin ax \text{ or } \frac{1}{\phi(D)} \cdot x^m \cos ax$$

Now $\dfrac{1}{\phi(D)}\left[x^m \cos ax + i \sin ax\right] = \dfrac{1}{\phi(D)} x^m e^{iax}$

$$= e^{iax} \cdot \frac{1}{\phi(D + ai)} x^n$$

RHS can be evaluated using the case 4 and then equating the real or imaginary parts, we get the required PI.

Case 7:

Suppose $F(x) = xv$ where v is any fuctions of x

$$D(xv) = x\,Dv + v$$
$$D^2(xv) = D(xDv + v)$$
$$|||^{ly}\ D^3(xv) = x\,D^3 v + 3D^2 v$$

In general

$$D^n(xv) = x\,D^n v + n\,D^{n-1} v$$
$$\text{or } D^n(xv) = x\,D^n v + \frac{d}{dD}(D^n)v$$

Since $\phi(D)$ is a polynomial in D, we may write the above result as

$$\phi(D)(xv) = x\,\phi(D)v + \phi'(D)v \quad \text{...I}$$

Now put $\phi'(D)v = V_1$ so that

$$V_0 = \frac{1}{\phi(D)} V_1 \text{ in equation I.}$$

Then $\phi(D)\left[x\dfrac{1}{\phi(D)} V_1\right] = xV_1 + \phi'(D)\dfrac{1}{\phi(D)} V_1$ ...II

Operating on both sides of II by $\dfrac{1}{\phi(D)}$ we get

$$x\frac{1}{\phi(D)} V_1 = \frac{1}{\phi(D)}(x V_1) + \frac{1}{\phi(D)}\phi'(D)\frac{1}{\phi(D)} V_1$$

This implies that

$$\frac{1}{\phi(D)} D(V_1) = \left[x - \frac{1}{\phi(D)}\phi'(D)\right]\frac{1}{\phi(D)} V_1$$

Here V_1 is any function of x and hence we have

$$\frac{1}{\phi(D)}(xV) = \left[x - \frac{1}{\phi(D)}\phi'(D)\right]\frac{1}{\phi(D)} V$$

Example 1

Solve $(D^2 - 3D + 2)y = e^{5x} + 2$

Solution :

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$(m - 1)(m - 2) = 0$$
$$\therefore m = 1, 2.$$

$\therefore$ the complementary function is $y = Ae^x + Be^{2x}$

$$PI = \frac{1}{D^2 - 3D + 2} e^{5x} + \frac{1}{D^2 - 3D + 2} \cdot 2$$
$$= \frac{1}{25 - 15 + 2} e^{5x} + \frac{1}{2} 2$$
$$= \frac{e^{5x}}{12} + 1$$

$\therefore$ the complete solution is $y = Ae^{x} + Be^{2x} + \frac{e^{5x}}{12} + 1.$

Example 2;

Solve $(D^2 - D - 2)\, y = e^{2x} + e^{x}$

Solution :

The auxiliary equation is $m^2 - m - 2 = 0$

$\therefore m = 2, -1$

$\therefore$ the complementany function is $y = Ae^{2x} + Be^{-x}$

$$PI = \frac{1}{D^2 - D - 2} e^{2x} + \frac{1}{D^2 - D - 2} e^{x}$$

$$= \frac{1}{(D-2)(D+1)} e^{2x} + \frac{1}{1-1-2} e^{x}$$

$$= \frac{1}{3} \cdot \frac{1}{D-2} e^{2x} - \frac{1}{2} e^{x}$$

$$= \frac{1}{3} x\, e^{2x} - \frac{1}{2} e^{x}$$

$\therefore$ the complete solution is

$$y = Ae^{2x} + B\, e^{x} + \frac{1}{3} x\, e^{2x} - \frac{1}{2} e^{x}$$

Example 3

Solve $(D^2 - 6D + 9)\, y = e^{3x}$

Solution :

the auxiliary equation is $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0 \;\therefore m = 3, 3.$

the complementary function is

$$y = e^{3x} (Ax + B)$$

$$PI = \frac{1}{(D^2 - 6D + 9)} e^{3x} \qquad \text{(failure case)}$$

$$= \frac{1}{(D-3)^2} e^{3x}$$

$$= \frac{x^2}{2} e^{3x}$$

$\therefore$ the complete solution is

$$y = e^{3x} (Ax + B) + \frac{x^2}{2} e^{3x}$$

Example 4

Solve $(D^2 + D - 2)\, y = \sin 2x$

Solution :

the auxiliary equation is $m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$\therefore m = -2, 1$

the complementary function is

$$y = Ae^{x} + Be^{-2x}$$

$$PI = \frac{1}{D^2 + D - 2} \sin 2x$$

$$= \frac{1}{-4 + D - 2} \sin 2x$$

$$= \frac{1}{D - 6} \sin 2x$$

$$= \frac{D + 6}{D^2 - 36} \sin 2x$$

$$= \frac{(D + 6)}{-40} \sin 2x$$

$$= -\frac{1}{40} (2 \cos 2x + 6 \sin 2x)$$

$$= -\frac{1}{20} (\cos 2x + 3 \sin 2x)$$

$\therefore$ the complete solution is

$$y = Ae^{x} + B^{-2x} - \frac{1}{20} (\cos 2x + 3\sin 2x)$$

Example 5

Solve $(D^2 - 4D - 12)\, y = \sin x \sin 2x$

Solution :

$$(D^2 - 4D - 12)\, y = \frac{1}{2} [\cos x - \cos 3x]$$

the auxilliary equation is

$$m^2 - 4m - 12 = 0$$

$$(m - 6)(m + 2) = 0$$

the complementary function is $y = Ae^{6x} + Be^{-2x}$

$$PI = \frac{1}{D^2 - 4D - 12}\,\frac{1}{2}(\cos x - \cos 3x)$$

$$= \frac{1}{D^2 - 4D - 12}\,\frac{1}{2}\cos x - \frac{1}{D^2 - 4D - 12}\,\frac{1}{2}\cos 3x$$

$$= \frac{1}{-(4D + 11)}\,\frac{1}{2}\cos x - \frac{1}{-(4D + 21)}\,\frac{1}{2}\cos 3x$$

$$= -\frac{4D - 11}{(16D^2 - 121)}\,\frac{1}{2}\cos x + \frac{4D - 21}{16D^2 - 441}\,\frac{1}{2}\cos 3x$$

$$= \frac{1}{137}(4D - 11)\,\frac{1}{2}\cos x - \frac{(4D - 21)}{585}\,\frac{1}{2}\cos 3x$$

$$= \frac{1}{274}(-4\sin x - 11\cos x) - \frac{(-12\sin 3x - 21\cos 3x)}{1170}$$

$$= -\frac{4\sin x + 11\cos x}{274} + \frac{4\sin 3x + 7\cos 3x}{390}$$

$\therefore$ the complementary function is

$$y = Ae^{6x} + Be^{-2x}$$

$$- \frac{4\sin x + 11\cos x}{274} + \frac{(4\sin 3x + 7\cos 3x)}{390}$$

Example 6

Solve $(D^2 + a^2)\,y = \sin ax + a\cos ax$

Solution :

the auxiliary equation is $m^2 + a^2 = 0$

$$\therefore m = \pm a$$

the complementary function is

$$y = A\cos ax + B\sin ax$$

$$PI = \frac{1}{D^2 + a^2}\sin ax + \frac{1}{D^2 + a^2}a\cos ax$$

$$= IP \text{ of } \frac{1}{(D + ai)(D - ai)}e^{iax} + RP \text{ of } \frac{1}{(D + ai)(D - ai)}ae^{iax}$$

$$= IP \text{ of } \frac{1}{2ai}x\,e^{iax} + \frac{1}{2ai}xae^{iax}$$

$$= IP \text{ of } -\frac{ix}{2a}(\cos ax + i\sin ax)$$

$$+ RP \text{ of } \frac{xa}{2ai}(\cos ax + i\sin ax)$$

$$= -\frac{x\cos ax}{2a} + \frac{x\sin ax}{2}$$

$\therefore$ the complete solution is

$$y = A\cos ax + B\sin ax - \frac{x\cos ax}{2a} + \frac{x\sin ax}{2}$$

Exercise 1

Solve the following differential equations:

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$
2. $\frac{d^2y}{dx^2} = \frac{dy}{dx}$
3. $\frac{d^2y}{dx^2} + y = 0$ gives $y = 0$ for $x = 0$ and $y = -2$ for $x = \frac{\pi}{2}$
4. $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\,y = 0$
5. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$
6. $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2 + 3$
7. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^{3x} + 4\cos 3x$
8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 8\cos 2x$
9. $(D^2 - 4D + 3)\,y = \sin 3x\cos 2x$
10. $(D^2 + D + 2)\,y = e^x + \cos x$

11. $(D^2 + 4D - 5)\, y = e^{3x} + 4 \cos 4x$
12. $(D^2 + 1) = e^x + \sin x$
13. $(D^2 + 4)\, y = 7 + \cos 2x$
14. $(D^2 + 2D + 2)\, y = -2 \cos 2x - 4 \sin 2x$
    given that $y(0) = 1$ and $y'(0) = 1$
15. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 50x$ given $y = 0$, $\frac{dy}{dx} = 1$ when $x = 0$

Example 1

Solve $(D^2 + D + 1)\, y = x$

Solution :

the auxiliary equation is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ the complementary function is

$$y = e^{-\frac{1}{2}x}\left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{1}{D^2 + D + 1}x$$

$$= \left[1 + D + D^2\right]^{-1} x$$

$$= \left[1 - (D + D^2) + (D + D^2)^2 + \ldots\right] x$$

$$= \left[1 - D\right] x = x - 1$$

$\therefore$ the complete solution is

$$y + A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x + x - 1$$

Example 2

Solve $(D^2 + 4)\, y = x^2$

Solution :

the auxiliary equation is $m^2 + 4 = 0$

the complementary function is

$$y = A \cos 2x + B \sin 2x$$

$$PI = \frac{1}{D^2 + 4}x^2$$

$$= \frac{1}{4\left(1 + \frac{D^2}{4}\right)}x^2$$

$$= \frac{1}{4}\left[1 + \frac{D^2}{4}\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + \ldots\right] x^2$$

$$= \frac{1}{4}\left[x^2 - \frac{1}{2}\right]$$

$\therefore$ the complete solution is

$$y = A \cos 2x + B \sin 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)$$

Example 3

Solve $(D^2 - 2D + 1)\, y = x^2 + 1 + \sin 2x$

Solution :

the auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m - 1)^2 = 0 \text{ or } m = 1, 1$$

The complementary function is

$$y = e^x (Ax + B)$$

$$PI_1 = \frac{1}{D^2 - 2D + 1}(x^2 + 1)$$

$$= (1 - D)^{-2}(x^2 + 1)$$

$$= (1 + 2D + 3D^2 + \ldots)(x^2 + 1)$$

$$= (1 + 2D + 3D^2)(x^2 + 1)$$

$$= x^2 + 1 + 4x + 6$$

$$= x^2 + 4x + 7$$

$$PI_2 = \frac{1}{D^2 - 2D + 1} \sin 2x$$

$$= \frac{1}{-4 - 2D + 1} \sin 2x$$

$$= \frac{1}{-(2D+3)} \sin 2x$$

$$= -\frac{(2D-3)}{4D^2 - 9} \sin 2x$$

$$= -\frac{2D-3}{-25} \sin 2x$$

$$= \frac{1}{25}(4\cos 2x - 3\sin 2x)$$

∴ the complete solution is

$$y = e^x (Ax + B) + x^2 + 4x + 7 + \frac{1}{25}(4\cos 2x - 3\sin 2x)$$

Example 4

Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$

Solution :

The given DE is $(D^2 - D + 1)y = x^3 - 3x^2 + 1$

The auxiliary equations is

$$m^2 - m + 1 = 0$$

$$m = \frac{1 \pm i\sqrt{3}}{2}$$

∴ the complementary solution is

$$y = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{1}{(D^2 - D + 1)}\left(x^3 - 3x^2 + 1\right)$$

$$= \left[1 - (D - D^2)\right]^{-1}\left[x^3 - 3x^2 + 1\right]$$

$$= \left[1 + (D - D^2) + (D - D^2)^2 + (D - D^2)^3 + \ldots\ldots\right](x^3 - 3x^2 + 1)$$

$$= \left[1 + D - D^2 + D^2 - 2D^3 + D^3\right]\left[x^3 - 3x^2 + 1\right]$$

$$= \left(1 + D - D^3\right)\left(x^3 - 3x^2 + 1\right)$$

$$= x^3 - 3x^2 + 1 + 3x^2 - 6x - 6$$

$$= x^3 - 6x - 5$$

∴ the solution is $y = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + x^3 - 6x - 5.$

Example 5

Solve $(D^2 + 3D + 2)y = e^{2x} + x^2 + \sin x$

Solution :

The auxiliary equation is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$\therefore m = -1, -2.$$

The complementary function is $y = Ae^{-x} + Be^{-2x}$

$$PI_1 = \frac{1}{D^2 + 3D + 2} e^{2x}$$

$$= \frac{1}{4 + 6 + 2} e^{2x} = \frac{1}{12} e^{2x}$$

$$PI_2 = \frac{1}{D^2 + 3D + 2} x^2$$

$$= \frac{1}{2\left(1 + \frac{3D + D^2}{2}\right)} x^2$$

$$= \frac{1}{2}\left[1 + \frac{3D + D^2}{2}\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1 - \frac{(3D + D^2)}{2} + \frac{(3D + D^2)^2}{4} + \ldots\right] x^2$$

$$= \frac{1}{2}\left[1 - \frac{3D}{2} - \frac{D^2}{2} + \frac{9D^2}{4}\right] x^2$$

$$= \frac{1}{2}\left[1 - \frac{3D}{2} + \frac{7D^2}{4}\right]x^2$$

$$= \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$PI_3 = \frac{1}{D^2 + 3D + 2}\sin x$$

$$= \frac{1}{-1 + 3D + 2}\sin x$$

$$= \frac{1}{3D + 1}\sin x$$

$$= \frac{3D - 1}{9D^2 - 1}\sin x$$

$$= -\frac{1}{10}(3D - 1)\sin x$$

$$= -\frac{1}{10}(3\cos x - \sin x)$$

∴ the complete solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{12}e^{2x} + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right] - \frac{1}{10}\left[3\cos x - \sin x\right]$$

Exercise 2

1. $(D^2 - 2D + 1)\,y = x^3$
2. $(D^2 - 1)\,y = 2 + 5x$
3. $(D^2 - 2D + 1)\,y = x + 1$
4. $(D^2 + 9)\,y = x^2$
5. $(D^2 - D + 1)\,y = x^3 - 3x^2 + 1$
6. $(D^2 - D)\,y = 1 - 11x$
7. $(D^2 + D - 2)\,y = x^2 - 2x + 3$
8. $(D^2 - 6D + 9)\,y = x^2 + e^x$
9. $(D^2 - 4D)\,y = x^2 - 1$
10. $(D^2 - 2D - 3)\,y = 3x^2 - 5$
11. $(D^2 - 3)\,y = x^2 - e^x$
12. $(D^2 - 7D)\,y = x^2$

13. $(D^2 + 2D + 1)\,y = x^2 + 1 - e^x$
14. $(D^2 + 7D + 10)\,y = x^2 - 4 + e^x$ given $y(0) = 0, y'(0) = 2$
15. $(D^2 - 1)y = 2e^x + 3x$
16. $(D^2 + D - 2)\,y = x + \cos x$

Example 1

Solve $(D^2 - 4D + 3)\,y = x^3 e^{2x}$

Solution :

The auxiliary equation is $m^2 - 4m + 3 = 0$

$$(m - 1)(m - 3) = 0$$

$$\therefore m = 1, 3$$

The complementary function is $y = Ae^x + Be^{3x}$

$$PI = \frac{1}{D^2 - 4D + 3}x^3 e^{2x}$$

$$= e^{2x}\frac{1}{(D+2)^2 - 4(D+2) + 3}x^3$$

$$= e^{2x}\frac{1}{D^2 + 4D + 4 - 4D - 8 + 3}x^3$$

$$= e^{2x}\frac{1}{D^2 - 1}x^3$$

$$= -e^{2x}\frac{1}{1 - D^2}x^3$$

$$= -e^{2x}\left(1 - D^2\right)^{-1}x^3$$

$$= -e^{2x}\left(1 + D^2\right)x^3$$

$$= -e^{2x}\left(x^3 + 6x\right)$$

∴ the complete solution is $y = Ae^x + Be^{3x} - e^{2x}\left(x^3 + 6x\right)$

Example 2

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cos 2x$

Solution :

The auxiliary equation is $m^2 - 5m + 6 = 0$

$$(m - 2)(m - 3) = 0$$

$$m = 2, 3$$

The complementary functon is $y = Ae^{2x} + Be^{3x}$

$$PI = \frac{1}{D^2 - 5D + 6} . e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos 2x$$

$$= e^x \frac{1}{D^2 - 3D + 2} \cos 2x$$

$$= e^x \frac{1}{-4 - 3D + 2} \cos 2x$$

$$= -e^x \frac{1}{3D + 2} \cos 2x$$

$$= -e^x \frac{3D - 2}{9D^2 - 4} \cos 2x$$

$$= -e^x \frac{3D - 2}{-40} \cos 2x$$

$$= \frac{e^x}{40} (-6 \sin 2x - 2 \cos 2x)$$

$$= \frac{-e^x (3 \sin 2x + \cos 2x)}{20}$$

∴ the complete solution is

$$y = Ae^{2x} + Be^{3x} - \frac{e^x}{20} (3 \sin 2x + \cos 2x)$$

Example 3

Solve $\frac{d^2y}{dx^2} + 4y = x \sin x$

Solution :

$$(D^2 + 4) y = x \sin x$$

the auxiliary equation is $m^2 + 4 = 0$

i e $m = \pm 2i$

the complementary function is $y = A \cos 2x + B \sin 2x$

$$PI = \frac{1}{D^2 + 4} x \sin x$$

$$= \left[x - \frac{2D}{D^2 + 4} \right] \frac{1}{D^2 + 4} \sin x$$

$$= \left[x - \frac{2D}{D^2 + 4} \right] \frac{1}{-1 + 4} \sin x$$

$$= \frac{1}{3} \left[x - \frac{2D}{D^2 + 4} \right] \sin x$$

$$= \frac{1}{3} \left[x \sin x - \frac{2D}{-1 + 4} \sin x \right]$$

$$= \frac{1}{3} \left[x \sin x - \frac{2 \cos x}{3} \right]$$

$$= \frac{x \sin x}{3} - \frac{2 \cos x}{9}$$

∴ the complete solution is

$$y = A \cos 2x + B \sin 2x + \frac{x \sin x}{3} - \frac{2 \cos x}{9}$$

Example 4

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

Solution :

$$(D^2 - 3D + 2) y = xe^{3x} + \sin 2x$$

the auxiliary equation is $m^2 - 3m + 2 = 0$

$$(m - 1)(m - 2) = 0$$

$$\therefore m = 1, 2.$$

the complementary function is $y = Ae^x + Be^{2x}$

$$PI = \frac{1}{D^2 - 3D + 2} (x e^{3x} + \sin 2x)$$

$$= \frac{1}{D^2 - 3D + 2} x e^{3x} + \frac{1}{D^2 - 3D + 2} \sin 2x$$

$$= e^{3x} \frac{1}{(D+3)^2 - 3(D+3) + 2} x + \frac{1}{-2^2 - 3D + 2} \sin 2x$$

$$= e^{3x} \frac{1}{D^2 + 3D + 2} x - \frac{1}{2 + 3D} \sin 2x$$

$$= \frac{e^{3x}}{2} \cdot \frac{1}{1+\frac{3D+D^2}{2}} x - \frac{2-3D}{4-9D^2}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1+\frac{3D+D^2}{2}\right]^{-1} x - \frac{2-3D}{40}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1-\left(\frac{3D+D^2}{2}\right)\right]x - \frac{2\sin 2x - 6\cos 2x}{40}$$

$$= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) - \frac{\sin 2x - 3\cos 2x}{20}$$

∴ the complete solution is

$$y = Ae^{3x} + Be^{2x} + \frac{xe^{3x}}{2} - \frac{3e^{3x}}{4} - \frac{\sin 2x - 3\cos 2x}{20}$$

Example 5

Solve $(D^2+1)y = x^2\cos x$

Solution :

the auxiliary equation is $m^2+1=0$

$m = \pm 1.$

the complementary function is $y = A\cos x + B\sin x$

$$PI = \frac{1}{D^2+1}x^2\cos x$$

$$= RP \text{ of } \frac{1}{D^2+1}x^2 e^{ix}$$

$$= RP \text{ of } e^{ix}\frac{1}{(D+i)^2+1}x^2$$

$$= RP \text{ of } e^{ix}\frac{1}{D^2+2iD}x^2$$

$$= RP \text{ of } e^{ix}\frac{1}{2iD\left(1+\frac{D}{2i}\right)}x^2$$

$$= RP \text{ of } e^{ix}\left[-\frac{i}{2D}\left(1-\frac{iD}{2}\right)^{-1} x^2\right]$$

$$= RP \text{ of } e^{ix}\left[-\frac{i}{2D}\left(1+\frac{iD}{2}-\frac{D^2}{4}\right)x^2\right]$$

$$= RP \text{ of } -\frac{ie^{ix}}{2D}\left[x^2+ix-\frac{1}{2}\right]$$

$$= RP \text{ of } -\frac{ie^{ix}}{2}\left[\frac{x^3}{3}+i\frac{x^2}{2}-\frac{x}{2}\right]$$

$$= RP \text{ of } -\frac{i}{2}\left[\frac{x^3}{3}+\frac{ix}{2}-\frac{x}{2}\right](\cos x + i\sin x)$$

$$= -\frac{1}{2}\left[\left(\frac{x^3}{3}-\frac{x}{2}\right)(-\sin x) - \frac{x}{2}\cos x\right]$$

∴ the complete solution is

$$y = A\cos x + B\sin x + \frac{1}{2}\left[\left(\frac{x^3}{3}-\frac{x}{2}\right)\sin x + \frac{x}{2}\cos x\right]$$

Example 6

Solve $\left(D^2-2D+1\right)y = xe^x\sin x$

Solution :

the auxiliary equation is $m^2-2m+1=0$

$m = 1, 1$

∴ the complementary function is $y = e^x(Ax+B)$

$$PI = \frac{1}{D^2-2D+1}xe^x\sin x$$

$$= \frac{1}{(D-1)^2}xe^x\sin x$$

$$= e^x\frac{1}{(D+1-1)^2}x\sin x$$

$$= e^x\frac{1}{D^2}x\sin x$$

$$= e^x\left[x-\frac{2D}{D^2}\right]\frac{1}{D^2}\sin x$$

$$= e^x \left[x - \frac{2}{D}\right] (-\sin x)$$

$$= -e^x \left(x \sin x + 2\cos x\right)$$

$\therefore$ the complete solution is

$$y = e^x (Ax + B) - e^x \left(x \sin x + 2 \cos x\right)$$

Example 7

Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{1}{1+e^x}$

Solution :

the auxiliary equation $m^2 + m = 0$

$$m(m+1) = 0$$

$$\text{or} \quad m = 0, -1$$

the complementary function is $y = A + Be^{-x}$

$$PI = \frac{1}{D^2 + D} \frac{1}{1+e^x}$$

$$= \frac{1}{D(D+1)} \frac{1}{1+e^x}$$

$$= \left(\frac{1}{D} - \frac{1}{D+1}\right) \frac{1}{1+e^x}$$

$$= \frac{1}{D} \frac{1}{1+e^x} - \frac{1}{D+1} \cdot \frac{1}{1+e^x}$$

$$= \int \frac{1}{1+e^x} dx - e^{-x} \int e^x \frac{1}{1+e^x} dx$$

$$= \int \frac{1+e^x - e^x}{1+e^x} dx - e^{-x} \int \frac{e^x}{1+e^x} dx$$

$$= \int \left(1 - \frac{e^x}{1+e^x}\right) dx - e^{-x} \int \frac{e^x}{1+e^x} dx$$

$$= x - \log(1+e^x) - e^{-x} \log(1+e^x)$$

$\therefore$ the complete solution is

$$y = A + Be^x + x - \left(1 + e^{-x}\right) \log\left(1 + e^x\right)$$

Exercise 3

1. $(D^2 - 1)y = xe^x$
2. $(D^2 + 6D + 9)y = e^{-2x}(x+2)$
3. $(D^2 + 2D - 3)y = e^{2x}(1 + x^2)$
4. $(D^2 - 5D - 6)y = e^{2x}(1 + x)$
5. $(D^2 + 4D + 3)y = 8x\,e^x - 6$
6. $(D^2 - 2D + 1)y = x^2 e^x$
7. $(D^2 - D)y = 3x\,e^x$
8. $(D^2 + 2D + 1)y = x^2 e^{-x}$
9. $(D^2 - 3D + 2)y = 2e^{3x} \sin 2x$
10. $(D^2 - 4D + 3)y = 2x\,e^{3x} + 3e^x \cos 2x$
11. $(D^2 - D + 2)y = 58e^x \cos 3x$
12. $(D^2 - 1)y = \cosh x \cos x$
13. $(D^2 + 4D + 4)y = e^{-x} \sin 2x$
14. $(D^2 + 9)y = (x^2 + 1)e^{3x}$
15. $(D^2 + 4D + 3)y = e^x \cos 2x - \cos 3x$
16. $(D^2 - 2D + 4)y = e^x \sin x$
17. $(D^2 - 2D + 3)y = e^{-x} \sin x$
18. $(D^2 - 7D + 18)y = x^2 e^{2x}$
19. $(D^2 - 2D + 5)y = e^{2x} \sin x$
20. $(D^2 - 2D + 1)y = x^2 e^x$
21. $(D^2 + 2D + 1)y = e^{-x} x^2$
22. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$
23. $(D^2 - 2D + 5)y = e^{2x} \sin x$
24. $(D^2 + 4)y = x^2 \sin^2 x$

Answers

Exercise 1

1. $y = Ae^{2x} + Be^{4x}$
2. $y = 4 + Be^x$
3. $y = -2\sin x$
4. $y = e^{ax}[A\cos bx + B\sin bx]$
5. $y = Ae^{-x} + Be^{-3x} - \frac{1}{2}xe^{-3x}$
6. $y = Ae^{2x} + Be^{-2x} + \frac{1}{4}xe^{-3x} - \frac{2}{3}e^x - 1$