CHAPTER IX

Virtual Work

§ 1. Work: When a force acting on a body moves its point of application, it is said to do work on the body.

If the force is a constant, the work done by the force is defined as the product of the force and the distance through which the point of application moves in the direction of the force.

Let a constant force F acting on a body move its point of application from A to B in its direction.

Fig. 1.

The work done by the force = F × AB = F.s where AB = s

But suppose a force F acts along AB and the point of application moves from A to C, where AC is inclined at an angle θ to AB.

Draw CD $\perp$ to AB.

The resolved part of the displacement AC, along AB = AD.

$\therefore$ Work done by the force

= F.AD

= F.AC $\cos\theta$

= F.$s\cos\theta$ ($\because$ AC = s)

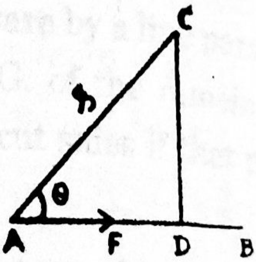

Fig. 2.

This can also be written as F $\cos\theta \times s$ and is equal to the product of the displacement s and the resolved part of F in the direction of the displacement.

Particular Cases:

(i) If $\theta = 0$, the displacement is in the direction of the force and work done = Fs.

(ii) If $\theta > 90°$, $\cos\theta$ is negative. The work done by the force is negative. We say that work is done against the force.

(iii) If $\theta = 90°$, $\cos\theta = 0$. In this case, the point of application is displaced in a direction at right angles to the direction of the force. Hence when *a force and its displacement are perpendicular, the work done by the force in such a displacement is zero.*

§ 2. Theorem: *The algebraic sum of the works done by a number of coplanar forces acting on a particle in any displacement of the particle is equal to the work done by their resultant.*

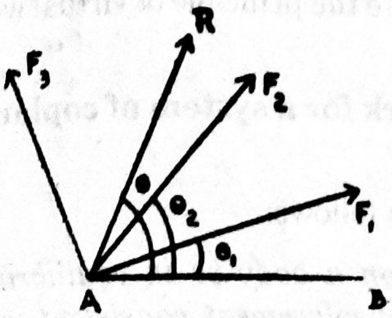

Fig. 3

Let forces $F_1, F_2, F_3...$ act on a particle at A and displace it to the position B. Let the directions of $F_1, F_2, F_3, ...$ make angles $\theta_1, \theta_2, \theta_3......$ with AB.

Let R be the resultant of the force, inclined at an angle θ to AB.

The algebraic sum of the works done by the forces

$= F_1\cos\theta_1.AB + F_2\cos\theta_2.AB + F_3\cos\theta_3.AB + ...$

$= AB(F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3 + ...)$

= AB (algebraic sum of the resolved parts of the forces along AB)

= AB. (resolved part of the resultant R along AB)

$= AB.R\cos\theta$

$= R\cos\theta.AB$

= Work done by the resultant.

§ 3. Method of Virtual work:

Consider a particle in equilibrium under the action of any number of forces. Suppose it is displaced in any direction while the forces remain constant in magnitude and direction.

By § 2, the total work done by the forces during the displacement is equal to the work done by the resultant. But as the particle is in equilibrium, the resultant is zero. Hence the algebraic sum of the works done by the forces during the displacement of the particle is zero.

It is often possible to deduce results in statical problems very easily by *imagining* a body at rest to be displaced through a small distance, finding the resulting small distances moved by the forces acting on it and equating the total work done by the forces to zero. Since the body in equilibrium is not actually displaced, the displacement given to it is called a *virtual* one and the work done by a force acting on the body during the displacement is called *virtual work.* 2m

In the next article, we shall state the principle of virtual work and prove it.

§ 4. Principle of virtual work for a system of coplanar forces acting on a body:

This principle can be stated as follows:

If a system of forces acting on a body be in equilibrium and the body undergoes a small displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works done by all the forces is zero; and, conversely, if this algebraic sum is zero, the forces will be in equilibrium.

We shall prove the above theorem for a system of coplanar forces.

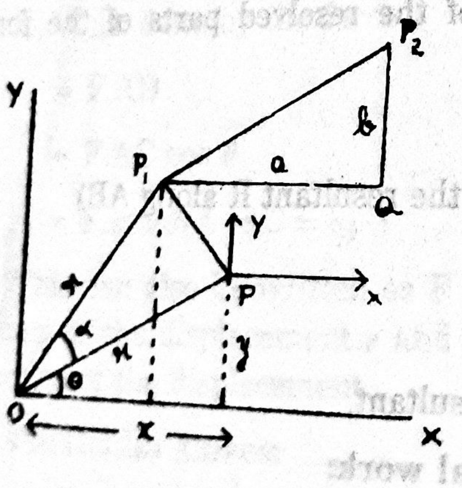

Fig. 4

Let OX, OY be any two perpendicular straight lines in the plane of the forces. Let F be one of the forces of the body, acting at P, whose coordinates referred to OX, OY are x and y and whose polar coordinates are r and θ. Then $x = r\cos\theta$ and $y = r\sin\theta$ where $OP = r$ and $\angle XOP = \theta$

Let the body be given a small displacement. This virtual displacement may be linear or angular or a combination of both. For a general case, we shall suppose that there is a small virtual angular displacement α, (by which P goes to P_1) and a linear displacement P_1P_2. The linear displacement may be supposed to be made up to two component displacements parallel to OX and OY. In the figure $P_1Q = a$ and $QP_2 = b$. Each point of the body will be treated in the same manner in the displacement. The quantities α, a, b are very small.

Now x - coordinate of P_2

$= $ x - coordinate of $P_1 + P_1Q$

$= r\cos(\theta + \alpha) + a$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha + a$

$= r\cos\theta\left(1 - \frac{\alpha^2}{2!} + ...\right) - r\sin\theta\left(\alpha - \frac{\alpha^3}{3!} + ...\right) + a$

$= r\cos\theta - r\sin\theta.\alpha + a$

(neglecting squares and higher powers of α which is small)

$= x - y\alpha + a = x + a - \alpha y$

∴ Change in the x coordinate of P $= a - \alpha y$

y coordinate of P_2

$=$ y coordinate of $P_1 + b$

$= r\sin(\theta + \alpha) + b$

$= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha + b$

$= r\sin\theta\left(1 - \frac{\alpha^2}{2!} + ...\right) + r\cos\theta\left(\alpha - \frac{\alpha^3}{3!} + ...\right) + b$

$= r\sin\theta + r\cos\theta\alpha + b$, omitting terms containing α^2, α^3 etc

$= y + x\alpha + b = y + b + \alpha x$

∴ Change in the y coordinate of P $= b + \alpha x$

Let the force F at P be resolved into two components X, Y parallel to the axes.

Then the virtual work done by F

$=$ Sum of the virtual works done by X and Y

$= X(a - \alpha y) + Y(b + \alpha x)$

$= aX + bY + \alpha(xY - yX)$

Similarly we have the virtual work done by each of the remaining forces of the system. It should be noted that the quantities a, b, α will be the same for all the forces while the components X, Y and the position (x, y) will vary. Hence the algebraic sum of the virtual works done by all the forces of the system

$$= \Sigma aX + \Sigma bY + \Sigma\alpha(xY - yX)$$

$$= a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) \quad ... \; ... \; ... \; ... \quad (1)$$

Now in the position in which the body was displaced, suppose the system of forces to be in equilibrium.

Then we have the following conditions of equilibrium.

(Refer § 10, page 168)

The sum of the components of the forces along OX and OY must be zero separately and the sum of the moments or the forces about O must be zero.

Hence $\Sigma X = 0 ... (i)$

$\Sigma Y = 0 ... (ii)$

and $\Sigma(xY - yX) = 0 ... (iii)$

Applying (i). (ii) and (iii) in (1), we have the result that the sum of the virtual works is equal to zero.)

Conversly *if the algebraic sum of the virtual works done by the forces is zero for any displacement, then the forces will be in equilibrium.*

As before, the sum of the virtual works for a general displacement is $= a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX)$ and this is given to be zero for all displacements.

i.e. $a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) = 0$ ... ... (2)

Give a displacement such that the body is displaced through a distance 'a' || OX only.

In this case, b = 0 and $\alpha = 0$

For these values, (2) becomes $a\Sigma X = 0$

As $a \neq 0$, this means that $\Sigma X = 0$ ... ... (i)

Similarly equation (2) will be satisfied when $a = 0$, $\alpha = 0$ and $b \neq 0$

For these values, (2) gives $b\Sigma Y = 0$

i.e. $\Sigma Y = 0$ ... (ii)

Finally, equation (2) will be satisfied when $a = 0$, $b = 0$ and $\alpha \neq 0$.

i.e. the body is given only an angular displacement.

For these values, from equation (2), we have

$\alpha.[\Sigma(xY - yX)] = 0$ ... ...

i.e. $\Sigma(xY - yX) = 0$ ... ...(iii)

(i),(ii) and (iii) are the usual conditions of equilibrium for a statical system.

As these are satisfied, the body must be in equilibrium.

Aliter: To prove the converse, the above argument may be simplified thus.

Since the equation $a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) \equiv 0,$

i.e. the equation is true for all values of a, b, and α, we must have coefficients of a, b, α to be zero each separately.

$\therefore$ $\Sigma X = 0, \Sigma Y = 0$ and $\Sigma(xY - yX) = 0$ which are the conditions for equilibrium.

Note: (i) The equation $\alpha\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) = 0$ is known as the *equation of virtual work.*

(ii) We have said that the total virtual work is zero, if the forces are in equilibrium To be more correct, we should say that *the sum of the virtual works is zero to* the first order of small quantities.

§ 5. Forces which may be omitted in forming the equation of virtual work:

There are certain forces which may be omitted in writing the equation of virtual work. We shall now consider such forces which commonly occur.

(i) *The tension of a string whose length does not alter during a small displacement of a system, does no work.*

Let AB be a string and T the tension at each of its ends A and B.

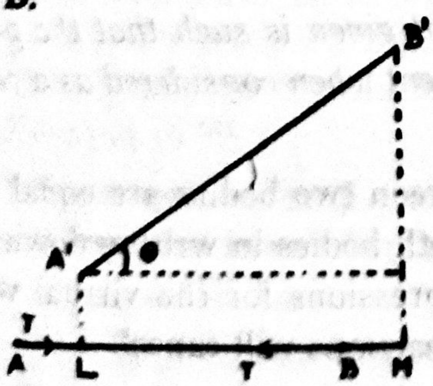

Fig. 5

Let the string be displaced through a small angle θ and take the position A'B'.

θ is assumed to be so small that its square and higher powers are neglected. Then, to the first power of θ, $\cos\theta = 1$.

Draw $A'L$ and $B'M \perp$ to AB.

Then the displacements or the points of application of the tensions at A and B in the directions of these tensions are AL and BM respectively.

Hence the sum of the works done by the tensions

$= T.AL - T.BM$

$= T(AL-BM)$

$= T(AL+LB-LB-BM)$

$= T(AB-LM)$

$= T(AB-A'B'.\cos\theta)$

$= T(AB-A'B')$ as $\cos\theta = 1$ approximately.

$= 0$. since $A'B' = AB$.

(ii) The reaction R of any smooth surface with which a body is in contact does no work, during a displacement of the body on the surface.

As the surface is smooth, the reaction R is entirely along the normal to the surface at the point of contact of the body and is therefore perpendiuclar to the displacement of this point. Hence the virtual work done by R is zero. This case includes the reaction of a smooth hinge.

(iii) The reaction at the point of contact of a body rolling on a fixed surface does no work during a small displacement.

In this case the body continues to roll on the surface, the point of contact is instantaneously at rest. Hence the normal reaction and frictional force at this point have zero displacements.

(iv) The mutual reactions between two bodies of a system do no work provided the displacement given is such that the point of contact gets the same displacement when considered as a point of either body.

The action and reaction between two bodies are equal and opposite. Hence if we consider both bodies in writing down the equation of virtual work, the expressions for the virtual work done by the equal and opposite reactions will cancel.

This does not apply to the case when there is a motion of the point of the two bodies and the bodies are rough e.g. a rough hinge.

§ 6. Work done by an extensible string:

Let AB (= l) be a string and T the tension at each of its ends A and B. Let the string be displaced through a small angle θ and come to the position $A'B'(= l + \Delta l)$. As θ is very small, $\cos\theta = 1$ nearly.

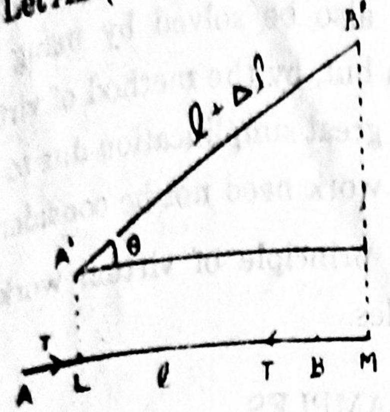

Fig. 6

Draw $A'L$ and $B'M \perp$ to AB.

Work done by the tension

$= T.AL - T.BM$

$= T(AL - BM)$

$= T(AL + LB - LB - BM)$

$= T(AB - LM)$

$= T(AB - A'B'\cos\theta)$

$= T(AB - A'B')$ as $\cos\theta = 1$ nearly

$= T(l - \overline{l + \Delta l})$

$= -T.\Delta l.$

Similarly the work done by the thrust in an extension of a rod from length l to $l + \Delta l$ is $+T.\Delta l$.. where T is the thrust in the rod.

Hence, in problems, It is not necessary to consider the separate virtual displacements of the ends of the string or the rod. It is enough to consider the increment in length only:

§ 7. Work done by the weight of a body.:

Let W be the weight of a body and y the height of its C.G, above some fixed level. In any small displacement, let y increase to $y + \Delta y$. Then the work done by W in this small displacement is $-W\Delta y$, because the displacement Δy of the C.G. is opposite in direction to W.

If, on the other hand, y measures the depth of the C.G. below a fixed level and in any displacement, y increases to $y + \Delta y$, work done by W is $+W.\Delta y$.

§ 8. Application of the principle of Virtual work:

The princeiple of virtual work is used to find the forces which hold a body or system of bodies in equilibrium, by equating to zero the work done in different virtual displacements. The principle will be especially useful in cases where we require the tension or thrust in a light spring or rod which is keeping a jointed framework. Such problems can also be solved by using the ordinary conditions of equilibrium but, by the method of virtual work, it can be seen that there is a great simplification due to the fact that these forces which do no work need not be considered.

The method of applying the principle of virtual work is illustrated in the following examples.

WORKED EXAMPLES

Ex. 1. *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freeely joined together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is W $\sqrt{3}$.*

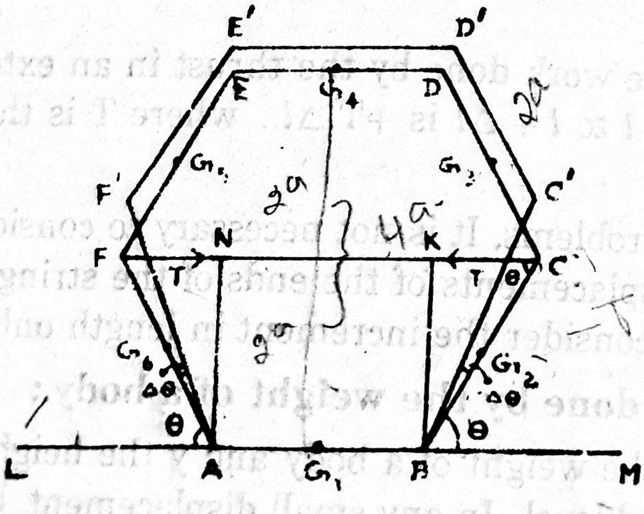

Fig. 7

Let $G_1, G_2, G_3, G_4, G_5, G_6$ be the middle points of the rods, W the weight and 2a the length of each rod and let T be the tension in the string.

Suppose $\angle FAL$ (or $\angle CBM$) $= \theta$

Give a small displacement to the system such that θ becomes $\theta + \Delta\theta$.

Due to this displacement, the positions of G_2, G_3, G_4, G_5, G_6 change but the lengths of the rods are not altered.

Since AB is fixed, we find, the heights of the midpoints of the other rods *above* AB.

The height of G_2 or G_6 above AB is $a\sin\theta$, that of G_3 or G_5 is $3a\sin\theta$ and the height of G_4 is $4a\sin\theta$.

Draw AN and BK $\perp$ to CF. $\cos\theta = \frac{FN}{AF} \rightarrow 2a\cos\theta = FN$

$FN = CK = 2a\cos\theta$.

$CK = 2a\cos\theta$

$\therefore$ The length of the string $= CF = FN + NK + KC$

$$= 4a\cos\theta + 2a$$

(x)

The work done by the weights at G_2 and G_6 is ↑ − ↓ +

$$= -2W\delta(a\sin\theta)$$

The work done by the weights at G_3 and G_5 is

$$= -2W\delta(3a\sin\theta)$$

The work down by the weight at $G_4 = -W\delta(4a\sin\theta)$

Since G_1 is not shifted, no work is done by the weight of AB.

By § 6, the work done by the tension T in the string CF is

$$= -T\delta(4a\cos\theta + 2a)$$

Hence the equation of virtual work is

$$-2W\delta(a\sin\theta) - 2W\delta(3a\sin\theta) - W\delta(4a\sin\theta)$$
$$-T\delta(4a\cos\theta + 2a) = 0.$$

i.e. $-2Wa\cos\delta\theta - 6Wa\cos\delta\theta - 4a\cos\theta\delta\theta$ (W)

$$+4aT\sin\theta\delta\theta = 0$$

i.e. $4a.\delta\theta(-3W\cos\theta + T\sin\theta) = 0$

Since $\delta\theta \neq 0$, $T\sin\theta - 3W\cos\theta = 0$.

or $T = 3W\cot\theta$ ... ...(1)

Now in the position of equilibrium, $\angle FAB = 120°$

$\therefore \theta = 60°$

Hence from (1),$T = 3W\cot 60°$

$$= \frac{3W}{\sqrt{3}} = W\sqrt{3}$$

Note:- It is very important to remember that all lengths of strings and heights of weights *must be expressed in terms of one variable only*. Also heights must be measured from a fixed level i.e. one which does not alter with the displacement we suppose the system to undergo.

Ex. 2. *Four rods, each of length a and weight W, are smoothly jointed together to form a rhombus ABCD, which is kept in shape by a light rod BD. The angle BAD is 60° and the rhombus is suspended in a vertical plane from A. Find the thrust in BD.* tension

[Hints:- vertical = sinθ, horizontal - cosθ]

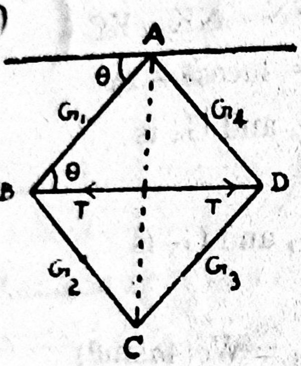

Fig. 8

Let G_1, G_2, G_3, G_4 be the midpoints of the rods. The combined C.G. (centre of gravity) of the rods is the point of intersection of the diagonals AC and BD.

Hence when the system is suspended from A, AC will be vertical and BD will be horizontal.

Let θ be the angle made by AB with the horizontal through A.

Give a small symmetrical displacement to the system such that θ becomes $\theta + \Delta\theta$.

We measure the depths of G_1, G_2, G_3, G_4 below A.

Depth of G_1 or G_4 below A $= \frac{a}{2}\sin\theta$

Depth of G_2 or G_3 below A $= \frac{3a}{2}\sin\theta$

Also $BD = 2a\cos\theta$.

The work done by the weights at G_1 and G_4

$= 2W.\delta\left(\frac{a}{2}\sin\theta\right)$

The work done by the weights at G_2 and G_3

$= 2W\,\delta\left(\frac{3a}{2}\sin\theta\right)$

Work done by the thrust T in BD $= T\delta(2a\cos\theta)$

Hence the equation of virtual work is

$$2W\delta\left(\frac{a}{2}\sin\theta\right) + 2W\delta\left(\frac{3a}{2}\sin\theta\right) + T\delta(2a\cos\theta) = 0$$

i.e. $Wa\cos\theta.\delta\theta + 3Wa\cos\theta.\theta - 2aT\sin\theta.\delta\theta = 0$

or $2a\delta\theta(2W\cos\theta - T\sin\theta) = 0$

Since $\delta\theta \neq 0, 2W\cos\theta - T\sin\theta = 0$

or $T = 2W\cot\theta$ ... ... (1)

Now in the position of equilibrium $\angle BAD = 60°$

$\therefore \angle BAC = 30°$ and $\theta = 60°$

Hence from (1), $T = 2W\cot 60° = \dfrac{2W}{\sqrt{3}}$

***Ex. 3** A solid hemisphere is supported by a string fixed to a point on its rim and to a point on the smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, Φ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by the principle of virtual work, that $\tan \Phi = \frac{3}{8} + \tan\theta$.*

(B.E.66)

Let O be the point of suspension, AB the base of the hemisphere, C its centre and G its centre of gravity, OA the string and D the point of contact of the hemisphere and the wall.

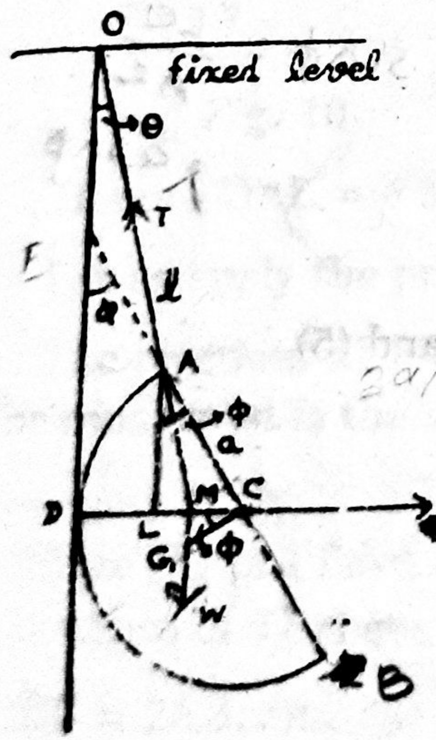

Fig. 9

Let OA $= l$ and CA $=$ a.

Then CG $= \frac{3a}{8}$ and $\angle ACG = 90°$

The forces acting on the hemisphere are (i) tension T along the string AO (ii) reaction R at D perpendicular to the wall and the sphere and hence along DC, the horizontal (iii) its weight W acting vertically downwards at G.

Let us apply the principle of virtual work, measuring the distance of G below the fixed point O.

The tension T and the reaction R will not do any work and hence they will not appear in the equation of virtual work.

The only force to be considered is W.

If y is the distance of G below O, the equation of virtual work is $W.\delta y = 0$. i.e. $\delta y = 0$ ... ... (1)

From the figure,

ΔOAE

$y = OE + ED + MG$

$\cos\theta = \frac{OE}{OA} \Rightarrow l\cos\theta$.

$= l\cos\theta + AL + CG.\sin\varphi$

$= l\cos\theta + a\cos\varphi + \frac{3a}{8}\sin\varphi$

ΔCMG. $\sin\phi = \frac{MG}{CG} \Rightarrow CG\sin\phi$

$\therefore \delta y = -l\sin\theta\delta\theta + a(-\sin\varphi + \frac{3}{8}\cos\varphi).\delta\varphi = 0$ ... (2)

Hence substituting (2) in (1),

ΔALC

$-l\sin\theta\delta\theta + a(-\sin\varphi + \frac{3}{8}\cos\varphi)\delta\varphi = 0$ (3)

$\cos\phi = \frac{AL}{AC} \Rightarrow a\cos\phi$

It must now be noted that the variables θ and φ are not independent. We can get a relation connecting them from the figure.

AE $= l\sin\theta$ from ΔOAE

$\sin\theta = \frac{AE}{OA} \Rightarrow l\sin\theta$.

and AE=DL=DC-LC=a $-$ $a\sin\varphi$

ΔALC

$\therefore l\sin\theta = a - a\sin\varphi$

$\sin\phi = \frac{LC}{AC} = a\sin\phi$

i.e. $l\sin\theta + a\sin\varphi = a$ ... ...(4)

Taking differentials of (4)

$l\cos\theta\delta\theta + a\cos\varphi.\delta\varphi = 0$ ... ...(5)

Now, eliminate $\delta\theta : \delta\varphi$ between (3) and (5).

From (3),

$$\frac{\delta\theta}{\delta\varphi} = \frac{-a(\sin\varphi + \frac{3}{8}\cos\varphi)}{l\sin\theta} \quad ... (6)$$

and from (5),

$$\frac{\delta\theta}{\delta\varphi} = -\frac{a\cos\varphi}{l\cos\theta} \quad ... ...(7)$$

Equating (6) and (7),

$$\frac{a(-\sin\varphi + \frac{3}{8}\cos\varphi)}{l\sin\theta} = -\frac{a\cos\varphi}{l\cos\theta}$$

or $$\frac{(-\sin\varphi + \frac{3}{8}\cos\varphi)}{\cos\varphi} = -\frac{\sin\theta}{\cos\theta}$$

i.e. $-\tan\varphi + \frac{3}{8} = -\tan\theta$

or $\tan\varphi = \frac{3}{8} + \tan\theta$.

Ex. 4. *An isosceles triangular lamina with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.*

Let ABC be the isosceles triangle resting between the pegs P and Q. Let PQ = a

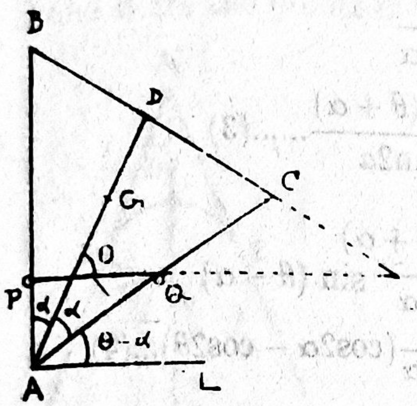

Fig. 10

Then BC= 3a

D is the midpoint of BC and C.G. is G on AD such that

AG:GD = 2:1

Let BC make an angle θ with the vertical.

As AD is $\perp$ to BC, it will make an angle θ with the horizontal.

Hence $\angle DAL = \theta$ and $\angle CAL = \theta - \alpha$

Let us apply the principle of virtual work.

The reactions at P and Q can be omitted and the only force to be considered is the weight W acting vertically downwards at G.

Since PQ is a fixed line (horizontal), we have to calculate y, the distance of G above PQ.

$BC = 2AB.\sin\alpha$ (the Δ being isosceles)

$\therefore AB = \dfrac{3a}{2\sin\alpha}$

$AB.\cos\alpha = AD$

$\therefore AD = \dfrac{3a\cos\alpha}{2\sin\alpha}$

$AG = \frac{2}{3}AD = \frac{2}{3}.\dfrac{3a\cos\alpha}{2\sin\alpha} = \dfrac{a\cos\alpha}{\sin\alpha}$

Height of G above A $= AG.\sin\theta$

$$= \frac{a\cos\alpha}{\sin\alpha}\sin\theta \quad \text{......(1)}$$

Height of Q above A = $AQ.\sin(\theta - \alpha)$ ... ... (2)

From Δ APQ, $\frac{AQ}{\sin\angle APQ} = \frac{PQ}{\sin 2\alpha}$

$\angle APQ = 180° - \angle PQA - \angle PAQ$

$= 180° - (\theta - \alpha) - 2\alpha = 180° - \theta - \alpha$

$$\therefore \frac{AQ}{\sin\{180° - (\theta + \alpha)\}} = \frac{PQ}{\sin 2\alpha}$$

$$\frac{AQ}{\sin(\theta + \alpha)} = \frac{PQ}{\sin 2\alpha}$$

$$\text{or } AQ = \frac{a\sin(\theta + \alpha)}{\sin 2a} \text{......(3)}$$

Hence using (3) in (2),

$$\text{height of Q above A} = \frac{a\sin(\theta + \alpha)}{\sin 2\alpha}\sin(\theta - \alpha)$$

$$= \frac{a}{2\sin 2\alpha}(\cos 2\alpha - \cos 2\theta) \text{...(4)}$$

y = Height of G above PQ

= Height of G above A – height of Q above A

$$= \frac{a\cos\alpha}{\sin\alpha}\sin\theta - \frac{a}{2\sin 2\alpha}(\cos 2\alpha - \cos 2\theta) \text{.........(5)}$$

Hence the equation of virtual work is

$-W.\delta y = 0$

i.e. $\delta y = 0$

Taking differentials of (5),

$$\delta y = \frac{a\cos\alpha}{\sin\alpha}.\cos\theta\delta\theta - \frac{a}{2\sin 2\alpha}(2\sin 2\theta)\delta\theta = 0$$

As $\delta\theta \neq 0$, we get

$$\frac{\cos\alpha\cos\theta}{\sin\alpha} - \frac{\sin 2\theta}{\sin 2\alpha} = 0$$

i.e. $\cos\alpha\cos\theta\sin 2\alpha - \sin\alpha\sin 2\theta = 0$

$2\cos\alpha\cos\theta\sin\alpha\cos\alpha - 2\sin\alpha\sin\theta\cos\theta = 0$

$2\sin\alpha\cos\theta(\cos^2\alpha - \sin\theta) = 0$

As $2\sin\alpha \neq 0$, either $\cos\theta = 0$ or $\cos^2\alpha - \sin\theta = 0$

If $\cos\theta = 0$, then $\theta = 90°$.

If $\cos^2\alpha - \sin\theta = 0$, then $\theta = \sin^{-1}(\cos^2\alpha)$

It may be noted that $\theta = 90°$ gives the symmetrical position of equilibrium and $\theta = \sin^{-1}(\cos^2\alpha)$ gives the oblique position of equilibrium

Ex. 5. *Two equal and light rods AB, AC each of length b are freely hinged at A and carry equal weights w at B and C. They rest on a smooth fixed sphere of radius a and centre O. Show that, in the position of equilibrium, the rods are inclined at an angle θ with the vertical given by $b\sin^3\theta = a\cos\theta$.*

D and E are the points of contact of the rods.

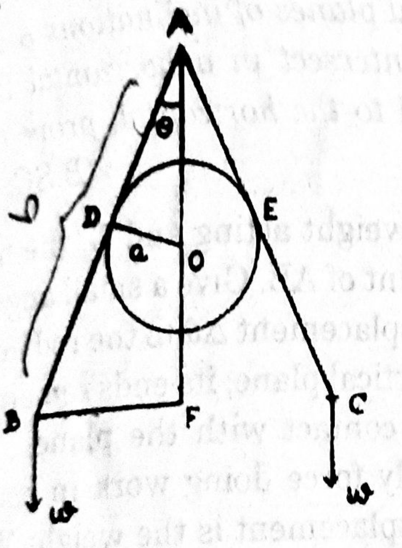

Fig. 11

In the position of equilibrium AO is vertical and BC is horizontal.

Give a small virtual displacement such that θ becomes $\theta + \Delta\theta$. The only forces that enter the eqution of virtual work are the two weights w.

Depth of B below O

$= OF = AF - AO = y$ (1)

From Δ ABF, $\cos\theta = \frac{AF}{AB} = \frac{AF}{b}$

$\therefore AF = b\cos\theta$ ... ... (2)

From Δ AOD, $\sin\theta = \frac{OD}{OA} = \frac{a}{OA}$

$\therefore OA = \frac{a}{\sin\theta}$ ... ...(3) (3)

Applying (2) and (3) in (1), we get

y = depth of B (or C) below O

$= b\cos\theta - \frac{a}{\sin\theta}$ ... ... (4)

The equation of virtual work is

$2w.\Delta y = 0$

i.e. $\Delta y = 0$.

From (4), $\Delta y = -b\sin\theta.\Delta\theta + \frac{a}{\sin^2\theta}\cos\theta\Delta\theta$

and this must be $= 0$

i.e. $\Delta\theta\left[-b\sin\theta + \frac{a\cos\theta}{\sin^2\theta}\right] = 0$

As $\Delta\theta \neq 0, -b\sin\theta + \frac{a\cos\theta}{\sin^2\theta} = 0$

i.e. $b\sin^3\theta = a\cos\theta$

Ex. 6. *A heavy uniform rod of length l rests with its ends in contact with two smooth inclined planes of inclinations α and β to the horizon. If the planes intersect in a horizontal line, and θ is the inclination of the rod to the horizontal, prove that* $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$ (B.SC. 78)

Let AB be the rod and W its weight acting and G, the mid-point of AB. Give a small angular displacement $\Delta\theta$ to the rod in the vertical plane, its ends remaining in contact with the plane. The only force doing work in such a displacement is the weight W.

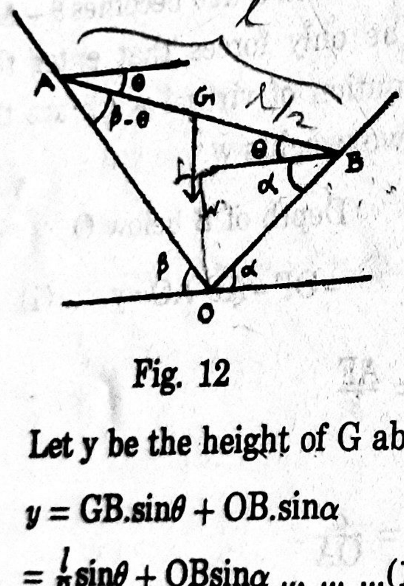

Fig. 12

The reactions at A and B do no work.

Let y be the height of G above O. $\sin\theta = \frac{GL}{GB}$

$y = GB.\sin\theta + OB.\sin\alpha$

$= \frac{l}{2}\sin\theta + OB\sin\alpha$ ... ...(1) $\sin\alpha = \frac{OL}{OB}$

We must express the variable length OB in terms of θ.

From Δ OAB,

$$\frac{OB}{\sin(\beta-\theta)} = \frac{AB}{\sin(\alpha+\beta)}$$

$\therefore OB = \frac{l\sin(\beta-\theta)}{\sin(\alpha+\beta)}$

Hence, from (1),

$$y = \frac{l}{2}\sin\theta + \frac{l\sin(\beta-\theta)}{\sin(\alpha+\beta)}\sin\alpha \; ... \; ... \; ... \; (2)$$

The equation of virtual work is

$-W.\Delta y = 0$

i.e. $\Delta y = 0$ ... ... (3)

Taking differentials of (2),

$$\Delta y = \frac{l}{2}\cos\theta.\delta\theta + \frac{l\sin\alpha}{\sin(\alpha+\beta)}.\cos(\beta-\theta).(-\Delta\theta)$$

$$= l.\delta\theta\left[\frac{\cos\theta}{2} - \frac{\sin\alpha\cos(\beta-\theta)}{\sin(\alpha+\beta)}\right]$$

Equating Δy to zero and noting that $\Delta\theta \neq 0$, we have

$$\frac{\cos\theta}{2} - \frac{\sin\alpha\cos(\beta-\theta)}{\sin(\alpha+\beta)} = 0$$

i.e. $\cos\theta(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

$$-2\sin\alpha(\cos\beta\cos\theta + \sin\beta\sin\theta) = 0$$

i.e. $\cos\theta(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = 2\sin\theta\sin\alpha\sin\beta$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{2\sin\alpha\sin\beta}$$

$$= \frac{1}{2}(\cot\alpha - \cot\beta)$$

Ex. 7. *A uniform ladder AB rests against a smooth wall at A and upon the ground at B. A boy whose weight is twice that of ladder climbs it. Prove that, the force of friction when he is at the top of the ladder is five times as great as when he is at the bottom.*

Let AB (= $2l$) be the ladder, G its centre of gravity, W its weight and θ its inclination to the vertical.

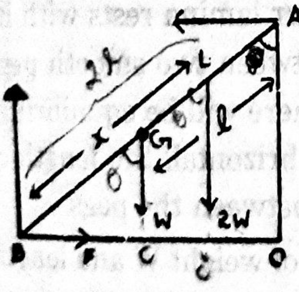

Fig. 13

Let F be the friction at the ground and L the position of the boy such that BL=x. BCG => cosθ = CG/BG

The height of G above the ground $= l\cos\theta$ and the height of L $= x\cos\theta$.

The distance of B from the foot of the wall $= BO = 2l\sin\theta$.

Let the ladder be slightly displaced so that it remains in contact with the wall and groundand θ changes by $\Delta\theta$.

The normal reactions at A and B do no work

I, ω, ...

Ground level

Hence the equation of virtual work is

$$-W.\delta(l\cos\theta) - 2W.\delta(x\cos\theta) - F.\delta(2l\sin\theta) = 0$$

i.e. $(Wl\sin\theta + 2Wx.\sin\theta - 2Fl\cos\theta)\delta\theta = 0$

As $\Delta\theta \neq 0$, the above gives $W\sin\theta(l + 2x) = 2Fl\cos\theta$

$$\therefore F = \frac{W}{2l}(l + 2x)\tan\theta \quad \ldots \ldots \ldots \quad (1)$$

Let F_1 and F_2 be the values of F when the boy is at the bottom and at the top respectively.

Putting $x = 0$ and $x = 2l$ in (1), we have

$$F_1 = \frac{W}{2l}.l\tan\theta = \frac{W\tan\theta}{2} \quad \ldots\ldots(2)$$

$$F_2 = \frac{W}{2l}.(l + 4l)\tan\theta = \frac{5W\tan\theta}{2} \quad \ldots\ldots(3)$$

Clearly $F_2 = 5F_1$.

Ex. 8. ***A step ladder in the form of the letter A with both the legs inclinedat an angle α to the vertical is placed on a horizontal floor and is held up by a cord connecting the middle point of its legs, there being no friction anywhere Prove that when a weight W is placed on one of the steps at a height equal to $\frac{1}{n}$ of the height of the ladder, the tension in the cord is increased by $\frac{W\tan\alpha}{n}$***

This has been already worked. Refer Ex.23 on page 199. G_1 and G_2 are the midpoints of the legs AB and AC.

Let AB= $2a$ and θ be the angle made by AB or AC with the vertical in any position. A weight W is attached at G_3 where $CG_3 = \frac{2a}{n}$.

Let T be the tension in the cord.

Give a small virtual displacement such that θ becomes $\theta + \Delta\theta$. The reactions at B and C and the reaction at the hinge at A do not do any work.

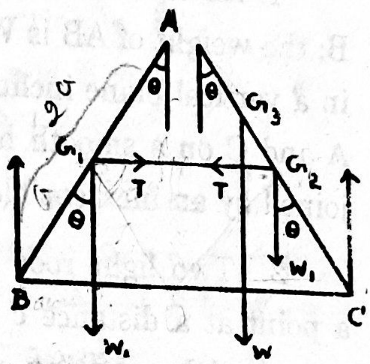

Fig. 14

Height of G_1 (or G_2) above BC $= a\cos\theta$

$\cos\theta = \frac{OG_1}{BG_1} = \frac{OG_1}{a}$

Height of G_3 above BC $= \frac{2a}{n}\cos\theta$.

$a\cos\theta = OG_1$.

$G_1G_2 = 2a\sin\theta$

Hence the equation of virtual work is

$$-2W_1\delta(a\cos\theta) - W\delta\left(\frac{2a}{n}\cos\theta\right) - T\delta(2a\sin\theta) = 0$$

i.e. $$2W_1 a\sin\theta.\Delta\theta + \frac{2aW}{n}\sin\theta.\Delta\theta - 2aT.\cos\theta.\Delta\theta = 0$$

$$\left[\left(W_1 + \frac{W}{n}\right)\sin\theta - T\cos\theta\right]\Delta\theta = 0$$

As $\Delta\theta \neq 0$, we hae $\left(W_1 + \frac{W}{n}\right)\sin\theta - T\cos\theta = 0$.

$$\therefore T = \left(W_1 + \frac{W}{n}\right) \tan\theta \ldots \ldots \ldots (1)$$

When there had been no W, let T_o be the tension.

Putting $W = 0$ in (1), $T_o = W_1 \tan\theta \ldots \ldots \ldots (2)$

Hence increase in tension $= T - T_o = \frac{W}{n}\tan\theta$

In equilibrium position, $\theta = \alpha$

and hence increase in tension $= \frac{W}{n}\tan\alpha$

Ex. 9. *A tripod consists of three equal uniform bars, each of length a and weight w, which are freely jointed at one extremity, their middle points being joined by strings of length b. The tripod is placed with its free ends in contact with a smooth horizontal plane and a weight W is attached to the common joint. Prove that the tension of each string is* $\frac{2}{3}(2W + 3w)\dfrac{b}{\sqrt{9a^2 - 12b^2}}$

Let OA, OB, OC be the bars.

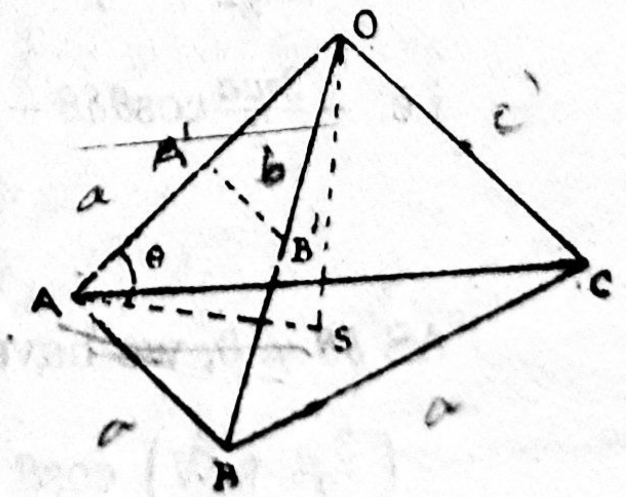

Fig. 15

ΔABC is equilateral and if S is its circumcentre, OS will be vertical.

Let $\angle OAS = \theta$ and A', B', C' be the midpoints of OA, OB, OC respectively.

The height of O above the plane ABC

$= OS = OA.\sin\theta = a\sin\theta$

The heights A', B', C' are $\frac{a}{2}\sin\theta$ each.

$SA = a\cos\theta$

From ΔABC,

$$\frac{AB}{\sin\angle ACB} = 2\times \text{ circumradius} = 2SA$$

i.e. $\frac{AB}{\sin 60^\circ} = 2a\cos\theta$ sin60° √3/2

$\therefore AB = 2a\cos\theta\sin 60^\circ = a\sqrt{3}\cos\theta$

$\therefore A'B' = \frac{1}{2}AB = \frac{a\sqrt{3}\cos\theta}{2}$ = length of each string.

Give a small symmetric displacement such that θ becomes $\theta + \Delta\theta$. Then O ascends and also the mid-points of the rods, while the strings are shortened by equal small amounts.

Let T be the tension in each string.

i) The work done by the weights of the bars

$$= -3w.\delta\left(\frac{a}{2}\sin\theta\right)$$

ii) The work done by the weight attached at O

$$= -W.\delta(a\sin\theta)$$

iii) The work done by the tensions in the three strings

$$= -3T\delta\left(\frac{a\sqrt{3}\cos\theta}{2}\right)$$

Hence the equation of virtual work is

$$-3w.\delta\left(\frac{a}{2}\sin\theta\right) - W.\delta(a\sin\theta) - 3T.\delta\left(\frac{a\sqrt{3}\cos\theta}{2}\right) = 0$$

i.e. $-\frac{3wa}{2}\cos\theta\delta\theta - Wa\cos\theta\delta\theta$

$$+\frac{3Ta\sqrt{3}}{2}\sin\theta\delta\theta = 0$$

AS $\delta\theta \neq 0$, we have

$$-\left(\frac{3w}{2} + W\right)\cos\theta + \frac{3T\sqrt{3}}{2}\sin\theta = 0$$

$$\therefore T = \frac{(3w+2W)}{3\sqrt{3}}\cot\theta \quad \ldots\ldots (1)$$

In the equilibrium position, $A'B' = b$

$\therefore \frac{a\sqrt{3}\cos\theta}{2} = b$ or $\cos\theta = \frac{2b}{a\sqrt{3}}$

cos θ / sin θ = cot θ

sin θ / cos θ = tan θ

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{4b^2}{3a^2}} = \frac{\sqrt{3a^2-4b^2}}{a\sqrt{3}}$$

$$\therefore \cot\theta = \frac{2b}{\sqrt{3a^2-3b^2}}$$

Hence substituting in (1), we have

$$T = \frac{(3w+2W)}{3\sqrt{3}}\,\frac{2b}{\sqrt{3a^2-4b^2}}$$

$$= \frac{2}{3}\,\frac{(3w+2W)b}{\sqrt{9a^2-12b^2}}$$

Ex. 10 *A uniform ladder of length l and weight W is held with its upper end resting against a smooth vertical wall and with its lower end on smooth horizontal surface: a man of weight W' stands on the ladder at a distance l' from the lower end. Show that if the ladder is kept from slipping by means of a couple, the moment of the couple is equal to $(\frac{1}{2}Wl + W'l')\sin\theta$. where θ is the inclination of the ladder to the vertical.*

(B.E. 55)

First, let us drive an expression for the work done by a couple.

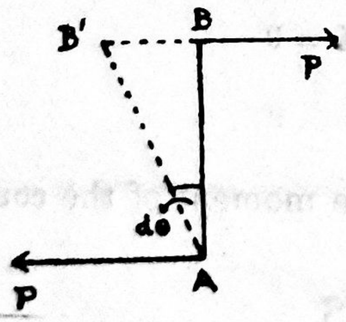

Fig. 16

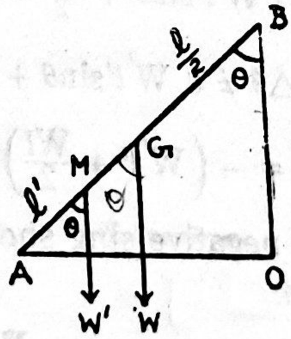

Fig. 17.

Let P, P be the forces of the couple acting at the ends of the arm AB. ~~Refer to fig. 16~~.

Clearly, if a translational displacement alone is given to the arm AB, the two forces will give equal but opposite amounts of work.

Hence work is done only when the arm AB rotates in the plane of the couple. Suppose the arm AB rotates about A through a small angle $d\theta$. Then B receives a small displacement $= BB' = AB\,d\theta$

Hence the work done by the couple

$= P.AB\ d\theta$

= the moment of the couple × the angle turned through.

In fig. 17, AB $(= l)$ is the ladder and W its weight acting vertically downwards at G, the midpoint of AB.

W is the weight of the man acting at L, where AL $= l'$.

AB makes an angle θ with the vertical.

Let L be the moment of the coupel required to prevent the ladder from slipping.

Give a small displacement such that θ increases by $\Delta\theta$.

The height of M above O $= l'\cos\theta$

The height of G above O $= \frac{l}{2}\cos\theta$

Hence the equation of virtual work is

$$-W'.\delta(l'\cos\theta) - W\delta\left(\frac{l}{2}\cos\theta\right) + L\delta\theta = 0$$

i.e. $\left(W'l'\sin\theta + \frac{Wl}{2}\sin\theta + L\right)\delta\theta = 0$

As $\Delta\theta \neq 0, W'l'\sin\theta + \frac{Wl\sin\theta}{2} + L = 0$

$\therefore L = -\left(W'l' + \frac{Wl}{2}\right)\sin\theta$

The negative sing shows that the moment of the couple is clockwise.