CHAPTER - II

Forces Acting at a Point

§ 1. Resultant and Components: Definition.

If two or more forces F_1, F_2, F_3,......... etc. act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces F_1, F_2, F_3,........ etc. put together, then the single force R is called the *resultant* of the forces F_1, F_2, F_3, ... etc. and the forces F_1, F_2, F_3 etc. are called the components of theforce R.

§ 2. Simple cases of finding the resultant:

If two forces P and Q act in the same direction simultaneously on a particle, the resultant is clearly equal to a force $P + Q$ acting in the same direction on it. If however P and Q act in opposite directions, their resultant is clearly equal to $P \sim Q$ and acts in direction of the greater force.

When two forces acting at a point are in different directions (i.e.) are inclined to each other, their resultant can be found with the help of a fundamental theorem in statics known as the *Law of the Parallelogram of Forces.*

§ 3. Parallelogram of Forces: Theorem

If two forces acting at a point be represented in magnitude and direction, by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.

Formal proofs of this law have been given by Bernoulli, D Alembert and Duchayla. The law can be verified experimentally. It is assumed here and taken as the fundamental principle of statics

Thus if the two forces P and Q acting at A are represented in magnitude and direction by the straight lines AB and AD and if the parallelogram DAB be completed, then the diagonal AC will represent in magnitude and direction the resultant of P and Q. In the language of vectors, the above law can be put as $\overline{AB} + \overline{AD} = \overline{AC}$.

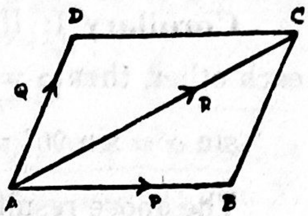


Fig. 1

Theory

4. Analytical expression for the resultant of two forces acting at a point:

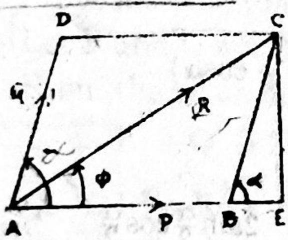


Fig. 2

Let the two forces P and Q acting at A be represented by AB and AD and let the angle between them be α.

i.e. $\boxed{\angle BAD = \alpha}$.

hyp opp adj

Complete the parallelogram BAD

Then the diagonal AC will represent the resultant (R)

Let R be the magnitude of the resultant and let it make an angle φ with P i.e. $\boxed{\angle CAB = \varphi}$

cab

Draw CE ⊥ to AB. BC = AD = Q.

From the right angled Δ CBE,

$$\sin \angle CBE = \frac{CE}{BC} \text{ i.e. } \sin \alpha = \frac{CE}{Q}$$

$$\therefore CE = Q \sin \alpha \ldots \ldots \ldots (i)$$

$$\cos \alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$$\therefore BE = Q \cos \alpha \ldots \ldots \ldots (ii)$$

$$\text{Now } R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$$

$$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$

$$= P^2 + 2PQ\cos\alpha + Q^2$$

$$\therefore R = \sqrt{(P^2 + 2PQ\cos\alpha + Q^2)} \ldots\ldots (1)$$

$$\text{Also } \tan\varphi = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha} \ldots\ldots (2)$$

φ 4

hypothenus

sinθ = opp/hyp

cosθ = adj/hyp

tanθ = opp/adj

(1) gives the magnitude and (2) the direction of the resultant in terms of P, Q and α.

Corollary 1: Soln: If the forces P and Q are at right angles to each other, then $\alpha = 90°$; $\cos\alpha = \cos 90° = 0$ and

$\sin\alpha = \sin 90° = 1$.

The above results become simpler and we have

magnitude $R = \sqrt{P^2 + Q^2}$ and resultant $\tan\varphi = \dfrac{Q}{P}$

These results may be easily inferred, since the parallelogram becomes a rectangle. Soln:

Corollary 2: If the forces are equal (i.e.) $Q = P$, then

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1+\cos\alpha)}$$

$$= \sqrt{2P^2 . 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

$$\text{and } \tan\varphi = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2}$$

Hence R: $\varphi =$

$$\text{i.e. } \varphi = \frac{\alpha}{2}$$

Thus the resultant of two equal forces P,P at an angle α is $2P\cos\frac{\alpha}{2}$ in a direction bisecting the angle between them.

This fact $\left(\text{that } \varphi = \frac{\alpha}{2}\right)$ is obvious otherwise, as the parallelogram becomes a rhombus.

statement:

Corollary 3: Let the magnitudes P and Q of two forces acting at an angle α be given.

Then their resultant R is greatest when $\cos\alpha$ is greatest.

Soln: i.e. when $\cos\alpha = 1$ or $\alpha = 0°$. $\alpha = \cos^{-1}(1)$ 0

In this case, the forces act along the same line in the same direction and $\boxed{R = P + Q.}$

The least value of R occurs when $\cos\alpha$ is least.

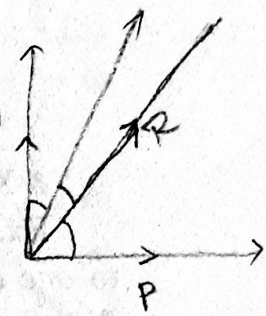
i.e. when $\cos\alpha = -1$ or $\alpha = 180°$. $\alpha = \cos^{-1}(-1) = \pi = 180$

In this case, the forces act along the same line but in opposite directions and $\boxed{R = P \sim Q}$

WORKED EXAMPLES

Ex. 1. *The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y. Prove that.*

$$P = (X^2 + QR)^{\frac{1}{2}} = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$



Prove also that, if $P + Q + R = 0, Y = X$

Soln: Let P and Q act at an angle α.

From the given data, we have the following results:

$$X^2 = P^2 + Q^2 + 2PQ\cos\alpha \ldots \quad \ldots \quad \ldots(1)$$

$$X^2 = P^2 + R^2 + 2PR\cos\alpha \ldots \quad \ldots \quad \ldots(2)$$

and $$Y^2 = Q^2 + R^2 + 2QR\cos\alpha \ldots \quad \ldots \quad \ldots(3)$$

(1) – (2) gives $0 = Q^2 - R^2 + 2P\cos\alpha(Q - R)$

i.e. $0 = (Q-R)(Q+R+2P\cos\alpha)$

But $Q \neq R$ and so $Q - R$ is $\neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

or $\cos\alpha = -\frac{(Q+R)}{2P} \ldots \quad \ldots \quad \ldots(4)$

Substituting (4) in (1), we have

$$X^2 = P^2 + Q^2 + 2PQ.-\left(\frac{Q+R}{2P}\right) = P^2 + Q^2 - Q^2 - QR$$

or $P^2 = X^2 + QR$ i.e. $P = (X^2 + QR)^{\frac{1}{2}}$

Substituting (4) in (3), we have

$$Y^2 = Q^2 + R^2 + 2QR.-\left(\frac{Q+R}{2P}\right)$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

or $$P = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

Q+R=-P

If P+Q+R=0, then Q+R= -P.

$\therefore$ From (4), $\cos\alpha = -\frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$

Putting $\cos\alpha = \frac{1}{2}$ in (2) and (3), we have

$$X^2 = P^2 + R^2 + PR \;\ldots\ldots\ldots (5)$$

and $Y^2 = Q^2 + R^2 + QR \;\ldots\ldots\ldots (6)$

(5) – (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$
$$= (P-Q)(P+Q+R)$$
$$= (P-Q).0$$
$$= 0$$

$\therefore X = Y.$

Relation between the

Ex. 2. *If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle φ with the direction of P, show that the resultant of forces (P + R) and Q acting at the same angle will make an angle $\frac{\varphi}{2}$ with the direction of P + R.* (B.Sc 84 M.K.U.)

First method:

Let $\overline{AB} = P$ and $\overline{AD} = Q$

From || gm ABCD, => P+Q=R

$\overline{AB} + \overline{AD} = \overline{AC} = R.$

To mark the force P + R, produce AB to E so that BE = AC.

In the || gm DAEF,

$\overline{AF}$ gives the new resultant

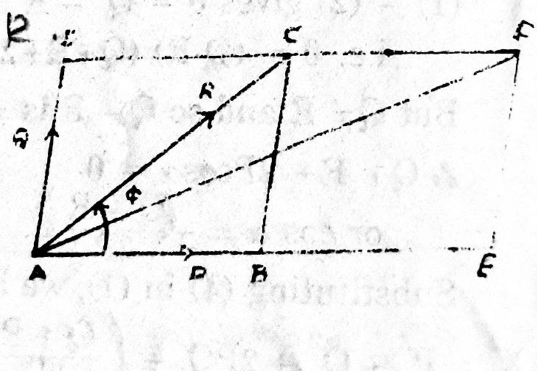


Fig. 3

In Δ CAF, CA = CF (each respresenting R in magnitude)

$\therefore \angle CAF = \angle CFA$

$= \angle FAE$ (alternate angles)

i.e. AF bisects $\angle CAB$

Second Method: The resultant of P+R and Q can be found in two stages. First, the resultant of P along AB and Q along AD is a force R along AC. Secondly, we have to find the resultant of the forces R along AC with an extra force R along AB. As these are equal, the final resultant bisects the angle BAC.

5. Triangle of Forces: A simple deduction from the parallelogram of forces is the following theorem, known as the *Triangle of forces.*

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

Theorem :-

parallelogram

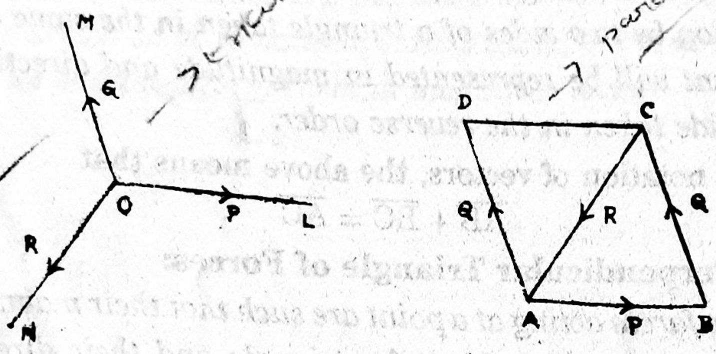


Fig.4

Let the forces, P,Q, R act at a point O and be represented in magnitude and direction by the sides AB,BC,CA, of the triangle ABC. We have to prove that they will be in equilibrium.

Complete the parallelogram BADC. As AD is equal and parallel to BC, AD also represents Q in magnitude and direction.

$$P + Q = \overline{AB} + \overline{AD}$$

$$= \overline{AC} \text{(by || gm law.)}$$

This shows that the resultant of the forces P and Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

pointer

Hence $P+Q+R=\overline{AC}$ at O + $\overline{CA}$ at O

$= \bar{0}$ (as the two vectors at O are equal and opposite)

∴ The forces are in equilibrium. 5m

Important note: In the above theorem, the forces P,Q,R are represented by the sides of the triangle ABC only in magnitude and direction but not in position. The forces *act at a point* and do not act along the sides of the triangle.

Corollary: From the proof of the above theorem, it is clear that the resultant of the forces represented in magnitude and direction by the two sides AB and BC of the triangle ABC, is represented in magnitude and direction by AC.

Note:-

This principle is stated as follows:

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

In the notation of vectors, the above means that

$$\overline{AB} + \overline{BC} = \overline{AC}$$

§ 6. Perpendicular Triangle of Forces:

If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.

Let the forces P,Q,R meet at O.

ABC is a triangle such that magnitudes of P,Q,R are proportional to the sides BC, CA and AB respectively of ΔABC and their directions are perpendicular to the corresponding sides all outwards.

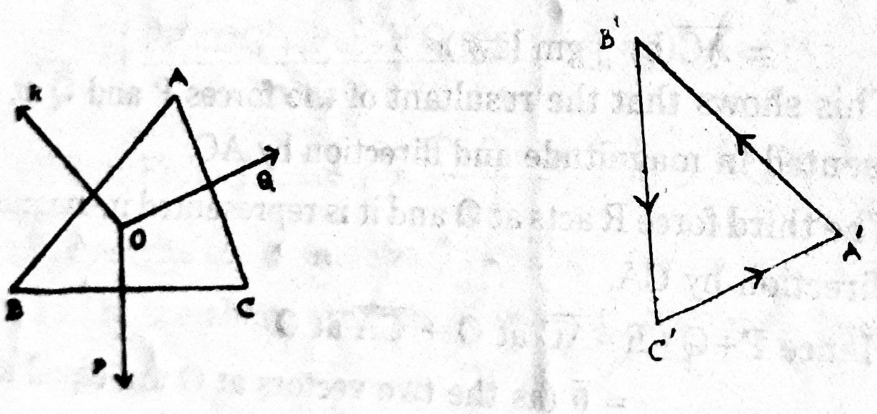


Fig. 5

We have to prove that they will be in equilibrium.

If we rotate the ΔABC through $90°$ in its own plane, we will get a new triangle $A'B'C'$ whose sides are parallel to the given forces and represent the forces both in magnitude and direction. Hence by the triangle of forces, P, Q, R are in equilibrium.

Note: The above result will also be true, if the directions of the forces, instead of being perpendicular to the corresponding sides, make equal angles in the same sense with them. The proof is exactly similar.

§ 7. Converse of the Triangle of Forces:

If three forces acting at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the directions of the forces shall represent them in magnitude also.

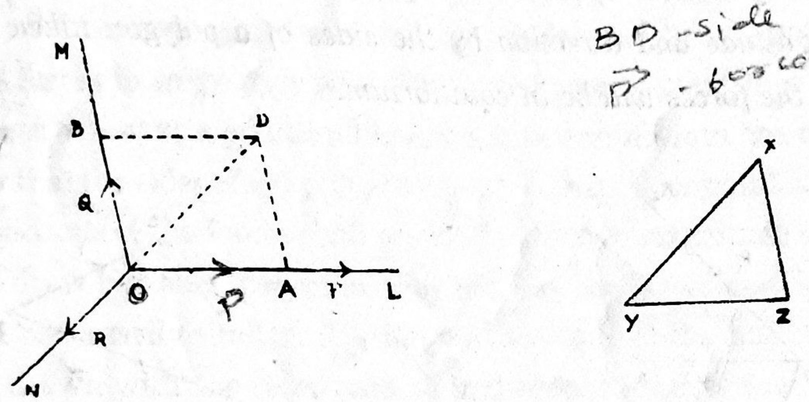


Fig. 6.

Let the three forces P,Q,R acting at O along the directions OL, OM and ON keep it in equilibrium. XYZ is a triangle such that the sides YZ,ZX and XY are parallel to the directions of P,Q,R respectively. We have to prove that the sides of ΔXYZ are proportional to the magnitudes of P,Q and R given that $P+Q+R=0$ (statically).

Along OL, cut off OA to represent the magnitude of P on some scale. i.e. let $\overline{OA}=P$.

On the same scale, make $\overline{OB}=Q$.

To get the resultant of P and Q, complete the || gm AOB.

The $P+Q=\overline{OA}+\overline{OB}=\overline{OD}$.

But $P+Q+R=\bar{0}$ (given) $\rightarrow R = -\overline{OD} = \overline{DO}$

i.e. $\overline{OD}+R=\bar{0}$ or $R=\overline{DO}$

This shows that the third force R is represented in magnitude on the same scale by DO and that DON is a straight line.

Hence the three forces P,Q and R are parallel and proportional to the sides of the triangle OAD.

Now any triangle like XYZ whose sides are parallel to the directions of P,Q and R will be similar to ΔOAD and hence

OA = P, OB = Q
DO = R

$$\frac{YZ}{OA} = \frac{ZX}{AD} = \frac{XY}{DO}$$

But $\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{DO}$; $\therefore \frac{YZ}{P} = \frac{ZX}{Q} = \frac{XY}{R}$

i.e. The sides of ΔXYZ will be proportional to P,Q,R.

Several number of forces

§ 8. The Polygon of Forces:

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

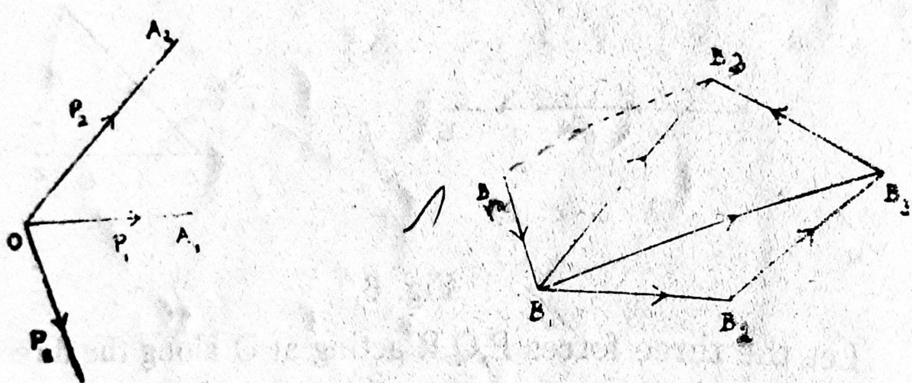


Fig. 7

Let the forces $P_1, P_2, \ldots \ldots P_n$ acting at O be represented in magnitude and direction by the sides $B_1B_2, B_2B_3, \ldots\ldots B_nB_1$ of the polygon $B_1B_2B_3\ldots\ldots B_n$.

We have to prove that the forces will be equilibrium. Compounding the forces by vector law, step by step, we have

$$P_1 + P_2 = \overline{B_1B_2} + \overline{B_2B_3} = \overline{B_1B_3}$$

$$P_1 + P_2 + P_3 = \overline{B_1B_3} + \overline{B_3B_4} = \overline{B_1B_4}$$

and $P_1 + P_2 + P_3 + \ldots\ldots + P_{n-1} = \overline{B_1B_{n-1}} + \overline{B_{n-1}B_n} = \overline{B_1B_n}$

It is to be noted that in each of the equations above, the resultant on the right side, of the forces named on the left side, acts at the point O.

The last force P_n is represented by $\overline{B_nB_1}$

$\therefore P_1 + P_2 + \ldots\ldots + P_{n-1} + P_n = \overline{B_1B_n}$ at O $+ \overline{B_nB_1}$ at O

$= \overline{0}$.

$\therefore$ The forces are in equilibrium.

[∵ sin(π − θ) = sin θ]

Note 1. The above theorem is true even when the forces acting at O are not in the same plane.

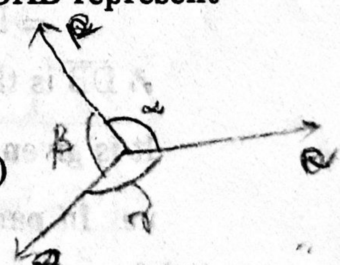
Note 2. *The converse of the Polygon of Forces is not true.* The converse of the triangle of forces is true because whenever the directions of three forces acting at a point and keeping it in equilibrium are known, all triangles drawn with their sides parallel to these directions, will be similar and hence represent the forces in mignitude also. But in the case of more than three forces acting at a point and keeping it in equilibrium, we cannot say that the sides of any polygon drawn with its sides parallel to the directions of the forces shall represent them in magnitude also. If we draw two such polygons, they will be merely equiangular and not necessarily similar. All that we can say is that a polygon can be drawn with the sides parallel and proportional to the forces.

§ 9. Lami's Theorem: Father Lami gave the converse of the triangle of forces in the following trigonometrical form:

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

Refer to the result and figure of § 7, page 15.

We have proved that the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.



Applying the sine rule to ΔOAD, we have

$$\frac{OA}{\sin\angle ODA} = \frac{AD}{\sin\angle DOA} = \frac{DO}{\sin\angle OAD} \ldots\ldots\ldots(1)$$

But $\angle ODA$ = alt. $\angle BOD = 180° - \angle MON$

$\therefore \sin\angle ODA = \sin(180° - \angle MON) = \sin\angle MON \ldots$ (2)

Also $\angle DOA = 180° - \angle NOL$

$\therefore \sin\angle DOA = \sin(180° - \angle NOL) = \sin\angle NOL \ldots$ (3)

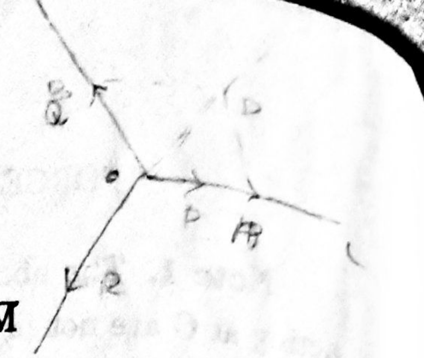
and $\angle OAD = 180° - \angle BOA = 180° - \angle LOM$

$\therefore \sin\angle OAD = \sin(180° - \angle LOM) = \sin\angle LOM \ldots$(4)

P/sin α = Q/sin β = R/sin γ

Substituting (2),(3),(4) in (1), we have

$$\frac{OA}{\sin\angle MON} = \frac{AD}{\sin\angle NOL} = \frac{DO}{\sin\angle LOM}$$



i.e. $\dfrac{P}{\sin\angle MON} = \dfrac{Q}{\sin\angle NOL} = \dfrac{R}{\sin\angle LOM}$

or $\dfrac{P}{\sin(Q,R)} = \dfrac{Q}{\sin(R,P)} = \dfrac{R}{\sin(P,Q)}$

WORKED EXAMPLES

Ex. 3. *Two forces act on a prticle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.* *(B.A.41)*

Let the forces P and Q acting at at A be represented in magnitude and direction by the lines AB and AD. Complete the parallelogram BAD.

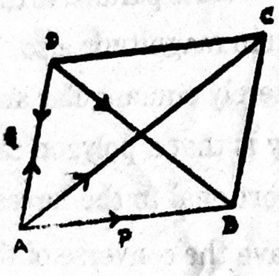


Fig. 8.

Then $P+Q=\overline{AB}+\overline{AD}=\overline{AC}$ (llgm law)

$\therefore$ $\overline{AC}$ is the sum of the two forces.

$P-Q=\overline{AB}-\overline{AD}$ ← interchanging

$=\overline{AB}+\overline{DA}$

$=\overline{DA}+\overline{AB}$

$=\overline{DB}$ (by triangle law)

$\therefore$ $\overline{DB}$ is the difference of the two forces.

It is given that $\overline{AC}$ and $\overline{DB}$ are at right angles.

i.e. In parallelogram ABCD, the diagonals AC and BD cut at right angles.

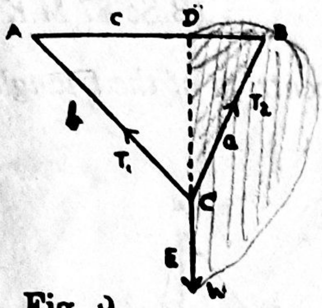
$\therefore$ ABCD must be a rhombus.

$\therefore$ AB = AD i.e. P = Q in magnitude.

Ex: 4. *A and B are two fixed points on a horizongal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio*

$$b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$$

Let T_1 and T_2 be the tensions along the strings CA and CB and W, the weight of the mass at C, acting vertically downwards along CE.



Produce EC to meet AB at D.

Since C is at rest under the action of the three forces, we have by Lami's theorem,

Fig. 9

$$\frac{T_1}{\sin\angle ECB} = \frac{T_2}{\sin\angle ECA} \quad \dots\dots\dots (1)$$

Now $\sin\angle ECB = \sin(180° - \angle DCB)$

$= \sin\angle DCB$

$= \sin(90° - \angle ABC) = \cos\angle ABC$ $= \cos B$

$\sin\angle ECA = \sin(180° - \angle ACD)$

$= \sin\angle ACD$

$= \sin(90° - \angle BAC) = \cos\angle BAC$ $= \cos B$

Hence (1) becomes

$$\frac{T_1}{\cos\angle ABC} = \frac{T_2}{\cos\angle BAC}$$

$$\therefore \frac{T_1}{T_2} = \frac{\cos\angle ABC}{\cos\angle BAC} = \frac{\cos B}{\cos A} \quad \dots\dots\dots (2)$$

In Δ ABC, we know that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence (2) becomes

$$\frac{T_1}{T_2} = \left(\frac{c^2 + a^2 - b^2}{2ca}\right) \times \left(\frac{2bc}{b^2 + c^2 - a^2}\right) = \frac{b(c^2 + a^2 - b^2)}{a(b^2 + c^2 - a^2)}$$

Ex.5 *ABC is a given triangle. Forces P,Q,R acting along the lines OA, OB, OC are in equilibrium. Prove that*

(i) $P:Q:R = a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2)$ *if O is the cicumcentre of the triangle.*

(B.Sc. 52 Madras; B.Sc. 87 Madurai)

(ii) $P:Q:R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$ *if O is the incentre of the triangle.*

(B.Sc. 63 M.U: B.Sc. 83 M.K.U.)

(iii) P:Q:R = a:b:c if O is the ortho centre of the triangle.

(B.Sc 87 M.K.U.)

(iv) P:Q:R = OA:OB:OC if O is the centroid of the triangle,

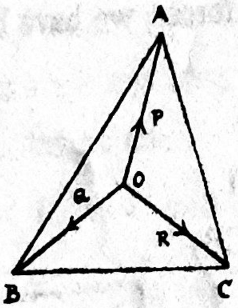


Fig.10

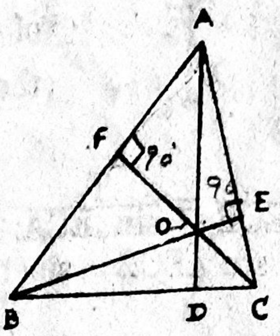


Fig. 11

ΔABC

$\Delta = \frac{1}{2} ab \sin C$

$\Delta = \frac{abc}{4R}$

By Lami's theorem, we have

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB} \quad \text{......(1)}$$

(i) When O is the circumcentre of the Δ ABC,

$\angle BOC = 2\angle BAC = 2A$; $\angle COA = 2B$ and $\angle AOB = 2C$

$\therefore$ (1) gives $\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$

$a^2 = b^2 + c^2 - 2bc$

i.e $\frac{P}{2\sin A\cos A} = \frac{Q}{2\sin B\cos B} = \frac{R}{2\sin C\cos C}$

... ... (2)

But $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ and $\sin A = \frac{2\Delta}{bc}$

where Δ is the area.

$\therefore 2\sin A\cos A = 2\,\frac{2\Delta}{bc}\,\frac{(b^2 + c^2 - a^2)}{2bc}$

$= \frac{2\Delta(b^2 + c^2 - a^2)}{b^2c^2}$

Similarly $2\sin B\cos B = \frac{2\Delta(c^2 + a^2 - b^2)}{c^2a^2}$

and $2\sin C\cos C = \frac{2\Delta(a^2 + b^2 - c^2)}{a^2b^2}$

So (2) becomes

$$\frac{P.b^2c^2}{2\Delta(b^2+c^2-a^2)} = \frac{Q.c^2a^2}{2\Delta(c^2+a^2-b^2)} = \frac{R.a^2b^2}{2\Delta(a^2+b^2-c^2)}$$

Multiplying throughout by $\frac{2\Delta}{a^2b^2c^2}$, we get

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

(ii) When O is the in-centre of the triangle,

OB and OC are the bisectors of $\angle B$ and $\angle C$.

$$\therefore \angle BOC = 180° - \frac{B}{2} - \frac{C}{2} = 180° - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180° - \left(90° - \frac{A}{2}\right) = 90° + \frac{A}{2}$$

Similarly $\angle COA = 90° + \frac{B}{2}$ and $\angle AOB = 90° + \frac{C}{2}$

So (1) becomes

$$\frac{P}{\sin\left(90° + \frac{A}{2}\right)} = \frac{Q}{\sin\left(90° + \frac{B}{2}\right)} = \frac{R}{\sin\left(90° + \frac{C}{2}\right)}$$

i.e. $\frac{P}{\cos\frac{A}{2}} = \frac{Q}{\cos\frac{B}{2}} = \frac{R}{\cos\frac{C}{2}}$

(iii) Let O be the ortho-centre of the triangle.

In fig. 11, AD, BE, CF are the altitudes.

Quadrilateral AFOE is cyclic. ($\because \angle AFO + \angle AEO$

$= 90° + 90° = 180°$)

$\therefore \angle FOE + A = 180°$ or $\angle FOE = 180° - A$

$\angle BOC =$ vertically opposite $\angle FOE = 180° - A$

Similarly $\angle COA = 180° - B$ and $\angle AOB = 180° - C$.

Hence (1) becomes

$$\frac{P}{\sin(180° - A)} = \frac{Q}{\sin(180° - B)} = \frac{R}{\sin(180° - C)}$$

i.e. $\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$

i.e. $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \left(\text{since } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$

(iv) When O is the centroid of the triangle,

we know that $\Delta BOC = \Delta COA = \Delta AOB$

and each $= \frac{1}{3}\Delta ABC$

$$\Delta BOC = \frac{1}{2}OB.OC \sin \angle BOC = \frac{1}{3}\Delta ABC$$

$$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3OB.OC}$$

Similarly $\sin \angle COA = \frac{2\Delta ABC}{3OC.OA}$ and $\sin \angle AOB = \frac{2\Delta ABC}{3OA.OB}$

Hence (1) becomes $\frac{P.3OB.OC}{2\Delta ABC} = \frac{Q.3OC.OA}{2\Delta ABC} = \frac{R.3OA.OB}{2\Delta ABC}$

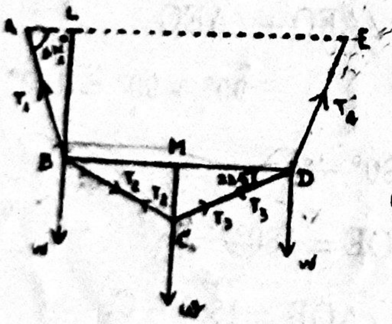
i.e. $P.OB.OC = Q.OC.OA = R.OA.OB$

Dividing by OA.OB.OC throughout,

$$\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$$

Ex. 6. *Weights W,w,W are attached to points B,C,D, respectively of a light string AE where B,C,D divide the string into 4 equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon with the ends A and E attached to points on the same level, show that*

$$W = (\sqrt{2}+1)w. \qquad \textit{(B.A.36; Andhra Uty.)}$$



ABCDE is a part of a regular octagon.

We know that each interior angle of a regular polygon of n sides

$$= \left(\frac{2n-4}{n}\right) \times 90°$$

Fig 12.

Putting $n = 8$, each interior angle of ABCDE

$$= \left(\frac{2\times 8-4}{8}\right) \times 90° = \frac{12}{8} \times 90° = 135°$$

Let the tensions in the portions AB,BC,CD,DE be T_1, T_2, T_3, T_4 respectively The string BC pulls B towards C and pulls C towards B, the tension being the same throughout its length. This fact is used to denote the forces acting at B, C and D.

In ΔBCD, $\angle BCD = 135°$

$$\therefore \angle CBD = \angle CDB = \frac{45°}{2} = 22\frac{1}{2}°$$

$\therefore \angle ABD = \angle ABC - \angle CBD = 135° - 22\frac{1}{2}° = 112\frac{1}{2}°$.

We know that every regular polygon is cyclic.

$\therefore$ A, B, C, D, E lie on the same circle.

$$\therefore \angle EAB = 180° - \angle BDE$$
$$= 180° - \{\angle CDE - \angle BDC\}$$
$$= 180° - \{135° - 22\frac{1}{2}°\}$$
$$= 45° + 22\frac{1}{2}° = 67\frac{1}{2}°$$

$\therefore \angle EAB + \angle ABD = 67\frac{1}{2}° + 112\frac{1}{2}° = 180°$.

$\therefore$ AE || BD.

$\therefore$ BD also is horizontal.

Let the vertical line through B meet AE at L and the vertical line through C meet BD at M.

Applying Lami's theorem for the three forces at B, we get

$$\frac{W}{\sin\angle ABC} = \frac{T_2}{\sin(180° - \angle ABL)}$$

i.e. $\dfrac{W}{\sin 135°} = \dfrac{T_2}{\sin\angle ABL} = \dfrac{T_2}{\sin 22\frac{1}{2}°}$.

($\because$ in rt. $\angle d\Delta ABL$, $\angle ABL = 90° - 67\frac{1}{2}°$)

$$\therefore T_2 = \frac{W}{\sin 135°}\sin 22\frac{1}{2}° \quad \ldots\ldots (1)$$

Similarly applying Lami's theorem for the three forces at C, we have $\dfrac{w}{\sin\angle BCD} = \dfrac{T_2}{\sin(180° - \angle MCD)}$

i.e. $\dfrac{w}{\sin\angle 135°} = \dfrac{T_2}{\sin\angle MCD} = \dfrac{T_2}{\sin(90° - 22\frac{1}{2}°)} = \dfrac{T_2}{\cos 22\frac{1}{2}°}$

$$\therefore T_2 = \frac{w}{\sin 135°}.\cos 22\frac{1}{2}° \quad \ldots\ldots (2)$$

Equating the two values of T_2 from (1) and (2), we have

$$\frac{W}{\sin 135°}\cdot\sin 22\frac{1}{2}° = \frac{w}{\sin 135°}\cdot\cos 22\frac{1}{2}°$$

i.e. $\dfrac{w}{W} = \tan 22\frac{1}{2}° = \sqrt{2} - 1$.

$$\therefore W = \frac{w}{\sqrt{2}-1} = \frac{w(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{w(\sqrt{2}+1)}{1}$$
$$= w(\sqrt{2}+1)$$

Ex. 7. ***A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If***

the slope of the plane be increased to β and the slope of the string unaltered, the tension of the string is doubled. Prove that $\cot\alpha - 2\cot\beta = \tan\gamma$ *(B.Sc. 72, 78, Madurai Uty.)*

P is the position of the weight. The forces acting at P are (i) its weight W downwards (ii) the normal reaction R perpendicular to the inclined plane and (iii) the tension T along the string at an angle γ to the horizontal. By Lami's theorem for the three forces at P,

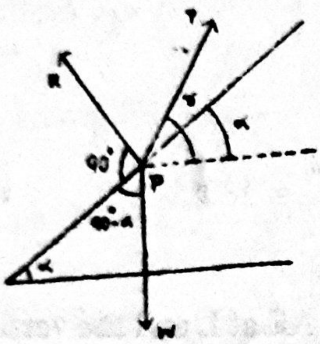


Fig. 13

$$\frac{T}{\sin(180^\circ - \alpha)} = \frac{W}{\sin[90^\circ - (\gamma - \alpha)]}$$

i.e. $\dfrac{T}{\sin\alpha} = \dfrac{W}{\cos(\gamma - \alpha)}$

$\therefore T = \dfrac{W\sin\alpha}{\cos(\gamma - \alpha)}$ ... ... (1)

In the second case, the inclination of the plane is β.

There is no change in γ.

If T_1 is the tension in the string, we will have

$T_1 = \dfrac{W\sin\beta}{\cos(\gamma - \beta)}$ ... ...(2)

But $T_1 = 2T$ (given)

$\therefore \dfrac{W\sin\beta}{\cos(\gamma-\beta)} = \dfrac{2W\sin\alpha}{\cos(\gamma - \alpha)}$

$\therefore \sin\beta\cos(\gamma - \alpha) = 2\sin\alpha\cos(\gamma - \beta)$.

i.e. $\sin\beta(\cos\gamma\cos\alpha + \sin\gamma\sin\alpha)$

$$= 2\sin\alpha(\cos\gamma\cos\beta + \sin\gamma\sin\beta).$$

$\sin\beta\cos\gamma\cos\alpha = 2\sin\alpha\cos\gamma\cos\beta + \sin\alpha\sin\beta\sin\gamma$

$$= \sin\alpha(2\cos\gamma\cos\beta + \sin\beta\sin\gamma)$$

$\therefore \dfrac{\cos\alpha}{\sin\alpha} = \dfrac{2\cos\gamma\cos\beta + \sin\beta\sin\gamma}{\sin\beta\cos\gamma}$

i.e. $\cot\alpha = 2\cot\beta + \tan\gamma$ or $\cot\alpha - 2\cot\beta = \tan\gamma$.

Ex. 8. *Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of* *the circle when the beads are in equilibrium on the upper half of the wire. Prove that the inclination of the string to the horizontal is given by*

$$\tan\alpha = \frac{w - w'}{w + w'}\tan\beta.$$

Answer

Let A and B be the beads of weights w and w' connected by a light string and sliding on a circular wire.

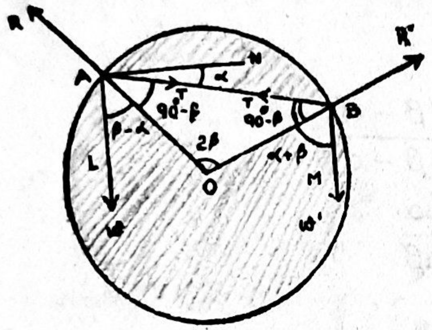


Fig. 14

In equilibrium position, $\angle AOB = 2\beta$. O being the centre of the circle.

$\therefore \boxed{\angle OAB = \angle OBA = 90° - \beta.}$

Let AB make an angle α with the horizontal AN.

AL and BM are the vertical lines through A and B.

$\angle OAL = 90° - \angle OAN = 90° - (\angle OAB + \angle NAB)$

$= 90° - (90° - \beta + \alpha) = \boxed{\beta - \alpha}$

Since AL || BM, $\angle ABM + \angle BAL = 180°$

$\therefore \angle ABM = 180° - \angle BAL = 180° - (90° - \alpha) = \boxed{90° + \alpha}$

$\therefore \angle OBM = \angle ABM - \angle ABO$

$= 90° + \alpha - (90° - \beta) = \boxed{\alpha + \beta}$

The forces acting on the bead w at A are

[i] weight w acting vertically downwards along AL

[ii] normal reaction R due to contact with the wire along the radius OA outwards.

and [iii] tension T in the string along AB.

Similarly the forces acting on the bead w' at B are

[i] weight w' acting vertically downwards along BM

[ii] normal reaction R' along the radius OB outwards

and [iii] tension T in the string along BA.

Applying Lami's theorem for the three forces at A,

$$\frac{w}{\sin[180° - (90° - \beta)]} = \frac{T}{\sin[180° - (\beta - \alpha)]}$$

i.e. $\dfrac{w}{\cos\beta} = \dfrac{T}{\sin(\beta - \alpha)}$ ...... (1)

Similarly applying Lami's theorem for the three forces at B,

$$\frac{w'}{\sin[180° - (90° - \beta)]} = \frac{T}{\sin[180° - (\beta + \alpha)]}$$

i.e. $$\frac{w'}{\cos\beta} = \frac{T}{\sin(\beta + \alpha)} \quad \dots \dots \quad (2)$$

Dividing (1) by (2) we have

$$\frac{w}{w'} = \frac{\sin(\beta + \alpha)}{\sin(\beta - \alpha)}$$

$$\therefore \frac{w - w'}{w + w'} = \frac{\sin(\beta + \alpha) - \sin(\beta - \alpha)}{\sin(\beta + \alpha) + \sin(\beta - \alpha)}$$

$$= \frac{2\cos\beta\sin\alpha}{2\sin\beta\cos\alpha} = \frac{\tan\alpha}{\tan\beta}$$

$$\therefore \tan\alpha = \left(\frac{w - w'}{w + w'}\right)\tan\beta$$

CHAPTER - III

Parallel Forces and Moments

1. Introduction: In the previous chapter we have considered the method of finding the resultant of two forces which meet at a point. We shall now consider how to find the resultant of two parallel forces. Such forces do not meet in a point and so we cannot find their resultant by direct application of the law of parallelogram of forces.

Two parallel forces are said to be *like* when they act in the same direction; they are said to be *unlike* when they act in opposite parallel directions.

2. To find the resultant of two like parallel forces acting on a rigid body:

Let like parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL. At A and B, introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented be AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

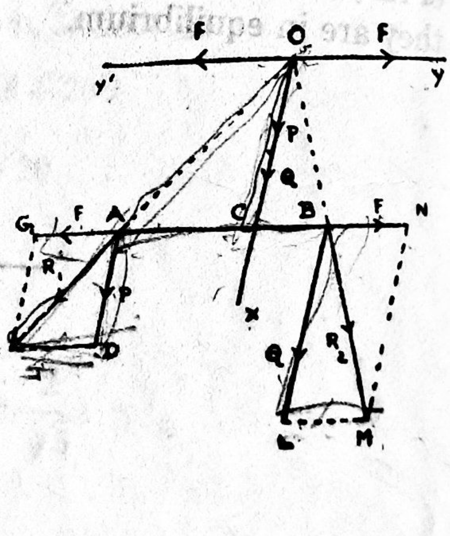


Fig 1

The two forces F and P acting at the point A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram ADEG. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At O draw Y'OY || to AB and OX || to the directions of P and Q. Reresolve R_1 and R_2 at O into their original components.

R_1 at O is equal to a force F along OY' and a force P along OX. R_2 at O is equal to a force F along OY and a force Q along OX. The two Fs at O cancel each other, being equal and opposite. We are now left with two forces P and Q acting along OX. Hence their resultant is a force (P+Q) acting along OX i.e. acting in a direction parallel to the original directions of P and Q.

Thus the magnitude of the resultant of two like parallel forces is their *sum*. The *direction* of the resultant is parallel to the components and in the same *sense*.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OAC and AED are similar.

$\therefore \frac{OC}{AD} = \frac{AC}{ED}$ i.e. $\frac{OC}{P} = \frac{AC}{F}$ or $F.OC = P.AC$ ...... (1)

Triangles OCB and BLM are similar.

$\therefore \frac{OC}{BL} = \frac{CB}{LM}$ i.e. $\frac{OC}{Q} = \frac{CB}{F}$ or $F.OC = Q.CB$ ...... (2)

From (1) and (2), we have $P.AC = Q.CB$

i.e. $\frac{AC}{CB} = \frac{Q}{P}$

i.e. the point C divides AB *internally* in the inverse ratio of the forces.

3. To find the resultant of two unlike and unequal parallel forces acting on a rigid body:

Let P and Q be two unequal and unlike parallel forces acting at the points A and B of the rigid body. Let P > Q and let them be represented by AD and BL. At A and B, introduce two equal and opposite forces F of arbitary magnitude along the line AB and let them be represented by AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

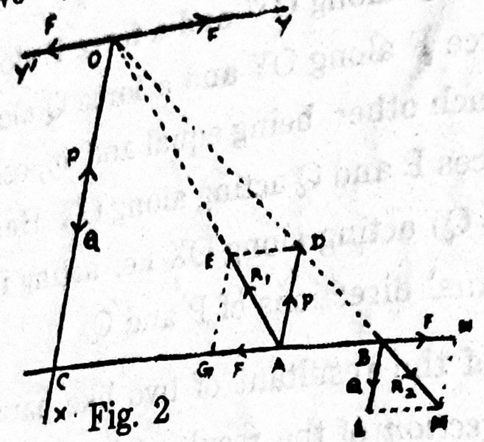


Fig. 2

The two forces F and P acting at A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram AGED. Similarly the two forces F and Q acting at B have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce AE and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At O, draw Y′OY || to AB and OX parallel to the directions of P and Q. Reresolve R_1 and R_2 at O into their original components. R_1 at O is equal to a force F along OY′ and a force P along XO. R_2 at O is equal to a force F along OY and a force Q along OX. The two Fs at O cancel each other, being equal and opposite. We are now left with a force P along XO and a force Q along OX. Clearly the resultant is a force P – Q (as P > Q) acting along XO i.e. acting in a direction parallel to that of P.

Thus the *magnitude* of the resultant of two unlike parallel forces is their *difference*. The direction of the resultant is parallel to and in the *sense of the greater component*.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OCA and EGA are similar.

$\therefore \frac{OC}{EG} = \frac{CA}{GA}$ i.e. $\frac{OC}{P} = \frac{CA}{F}$

$$\text{or } F.OC = P.CA \ \ldots\ldots (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM} \text{ i.e. } \frac{OC}{Q} = \frac{CB}{F} \quad \text{or } F.OC = Q.CB \ \ldots\ldots (2)$$

From (1) and (2), we have P.CA = Q.CB

$$\text{i.e. } \frac{CA}{CB} = \frac{Q}{P}$$

i.e. The point C divides AB *externally* in the inverse ratio of the forces.

Note: As P > Q, CB must be >CA.

Hence the resultant passes nearer the greater force.

Failure of the above construction:

The construction for finding the resultant of two unlike parallel forces P and Q will fail, if P = Q i.e. if the forces are equal in magnitude. In that case, in fig. 2, triangles AGE and BNM will be congruent.

$\angle GAE = \angle NBM$ and the lines AE and MB will be parallel.

There will be no such point as O.

Hence we conclude that *the effect of two equal and unlike parallel forces cannot be replaced by a single force.* Such a pair of forces have no single resultant and they constitute what is called a *couple*, which will be considered later on.

§ 4. Resultant of a number of parallel forces acting on a rigid body.

If a number of parallel forces P, Q, R, ... act on a rigid body, their resultant can be found by successive applications of § 2 and § 3. First, we find the resultant R_1 of P and Q; then we find the resultant R_2 of R_1 and R and this process is continued, until the final resultant is obtained. If the parallel forces are all like, the magnitute of the final resultant is the sum of the forces. If the parallel forces are not all like, the magnitute of the resultant is the algebraic sum of the forces each taken with its proper sign.

§ 5. **Conditions of equilibrium of three coplanar parallel forces:**

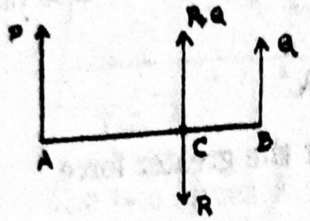


Fig. 3

Let P, Q, R be three forces parallel in one plane and be in equilibrium. Draw a line to meet the lines of action of these forces at A, B and C respectively.

If all the three forces are in the same sense, equilibrium will be clearly impossible. Hence two of them (say P and Q) must be like and the third R unlike.

The resultant of P and Q is (P + Q), parallel to P or Q and hence, for equilibrium, R must be equal and opposite to (P + Q).

$\therefore$ R = P + Q and the line of action of P + Q must pass through C.

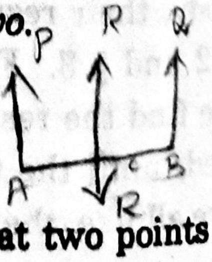
$\therefore$ P.AC = Q.CB

i.e. $\dfrac{P}{CB} = \dfrac{Q}{AC}$

and each $= \dfrac{P+Q}{CB+AC} = \dfrac{P+Q}{AB} = \dfrac{R}{AB}$

i.e. $\dfrac{P}{CB} = \dfrac{Q}{AC} = \dfrac{R}{AB}$

Thus, *if three parallel forces are in equilibrium, each is proportional to the distance between the other two.*



6. Centre of two parallel forces:

Let P and Q be two parallel forces acting at two points A and B. Then, their resultant R passes through a point C, which divides AB internally or externally in the ratio Q:P.

i.e. $\frac{AC}{CB} = \frac{Q}{P}$ ...... (1)

The position of C given by (1), depends only upon the positions of A and B and then magnitudes of the forces P and Q. It does not depend on the actual direction of P and Q. In other words, whatever be the common direction of parallelism of the forces P and Q, their resultant will always pass through a certain fixed point. This fixed point is called the *centre of the two parallel forces. Thus the centre of two parallel forces is a fixed point through which their resultant always passes whatever be the direction of parallelism.*

More generally, the resultant of a system of parallel forces of given magnitudes, acting at given points of a body, will always pass through a fixed point, for all directions of parallelism. This point is called the *centre of parallel forces.*

thing

WORKED EXAMPLES

Ex. 1. *Two men, one stronger than the other, have to remove a block of stone weighing 300 kgs. with a light pole whose length is 6 metre. The weaker man cannot carry more than 100 kgs. Where must the stone be fastened to the pole, so as just to allow him his full share of weight?*

Let A be the weaker man bearing 100 kgs., his full share of of the weight of the stone and B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC=x. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

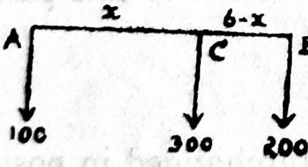


Fig. 4

∴ 100.AC = 200.BC

i.e. 100x = 200 (6-x) = 1200 - 200x

∴ 300x = 1200 or x=4.

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

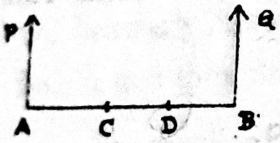
Ex. 2. *Two like parallel forces P and Q act on a rigid body at A and B respectively.*

(a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged. (B. Sc., 82 M.K.U.)

(b) If P and Q be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d, where $d = \frac{P-Q}{P+Q} \cdot AB$ (B. Sc.83 M.K.U.)

(a) Let C be the centre of two parallel forces with P at A and Q at B.

Then $P.AC = Q.CB$ ...... (1)



If Q is changed to $\frac{P^2}{Q}$, (P remaining the same), let D be the new centre of parallel forces.

Fig.5

Then $P.AD = \frac{P^2}{Q} DB$ ... ... (2)

i.e. $PQ.AD = P^2.DB$

or $Q.AD = P.DB$ ... ... (3)

Relation (3) shows that D is the centre of two like parallel forces, with Q and A and P at B.

(b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

$CD = d$.

From (3), $Q.(AC+CD) = P.(CB - CD)$

i.e. $Q.AC + Q.d = P.CB - P.d$

or $(Q + P).d = P.CB - Q.AC$

$= P(AB - AC) - Q(AB - CB)$

$= P.AB - P.AC - Q.AB + Q.CB$

$= (P - Q).AB[\because P.AC = Q.CB \text{from (1)}]$

$\boxed{\text{or } d = \frac{P-Q}{P+Q} \cdot AB}$ AB

Ex. 3. *Three like parallel forces, acting at the vertices of a triangle, have magnitudes proportional to the opposite sides. Show that their resultant passes through the incentre of the triangle.* (B. Sc., 81 M.K.U)

a, b, c is sides of a triangle

Let like parallel forces P, Q, R act at A, B, C.

It is given that $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$ ... (1)

7 Let the resultant of Q and R meet BC at D.

We know that the magnitude of the resultant is Q + R.

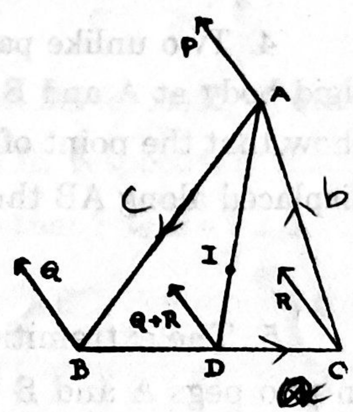


Fig. 6.

Also $\frac{BD}{DC} = \frac{\text{force at C}}{\text{force at B}} = \frac{R}{Q}$

$= \frac{c}{b}$ from (1)

$= \frac{AB}{AC}$

$\therefore$ AD is the internal bisector of $\triangle$ ABC

We have now to find the resultant of the two like parallel forces, Q + R at D and P at A.

Let this resultant meet AD at I.

Then $\frac{AI}{ID} = \frac{\text{force at D}}{\text{force at A}} = \frac{Q+R}{P}$

$= \frac{b+c}{a}$ from (1) ... ... (2)

From result (2), if it is clear that I is the incentre of the $\triangle$.

[If I is the incentre of $\triangle$ ABC and AD bisects $\angle$A internally, we have the result $\frac{AI}{ID} = \frac{b+c}{a}$.

For a proof of this result refer worked Ex. 13 on page 32].

§ 9. Geometrical Representation of a moment: 2m

13

Fig. 8

Let a force F acting on a body be represented in magnitude, direction and line of action by the line AB.

Let O be any given point and ON the perpendicular from O on AB or AB produced.

The moment of the force F about O

$= F \times ON = AB \times ON = 2\triangle\ AOB.$

Hence *if a force is represented completely by a straight line, its moment about any point is given by twice the area of the triangle which the straight line subtends at that point.*

§ 10. Sign of the moment : In fig.8, when the force F acts along AB, it will tend (continue) to rotate the lamina in the anticlockwise direction i.e. in a direction opposite to that in which the hands of clock move. In such cases, the moment is said to be *positive.* 2m
11 If the force tends to turn the body in a clockwise direction, its moment is said to be *negative.*

Thus the moment, of a force about a point has both magnitude and direction and is therefore a vector quantity.

§ 11. Unit of moment: The moment of a unit force about a point at a unit perpendicular distance from the line of action of the force is defined as the unit for the measurment of moments. If the unit of force be a poundal and unit of distance be one foot the unit of moment is a poundal-foot. If the unit of force be a
2 dyne and unit of distance be one centimetre, the unit of moment is a dyne-cm.

§ 12. Varigon's Theorem of Moments:

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

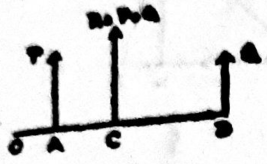


Fig. 9(a)

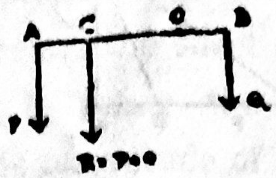


Fig. 9(b)

To prove this theorem, we consider two cases.

Case I: Let the forces be parallel.

Let P and Q be two parallel forces and O any point in their plane. Draw AOB perpendicular to the forces to meet their lines of action in A and B.

The resultant of P and Q is a force R (=P+Q) acting at C such that P.AC = Q.CB.

In fig 9(a),

the algebraic sum of the moments of P and Q about O

= P.OA + Q.OB

= P (OC – AC) + Q(OC+CB)

= (P + Q).OC – P.AC + Q.CB

= (P + Q).OC [$\because$ P.AC = Q.CB]

= R.OC

= moment of R about O

In fig. 9 (b), O is within AB and the algebraic sum of the moments of P and Q about O

= P.OA – Q.OB

= P (OC + CA) – Q (CB–CO)

= (P + Q).OC +P.CA-Q.CB

= (P + Q).OC [$\because$ P.AC = Q.CB]

= R.OC = moment of R about O

When the parallel forces P and Q are unlike and unequal, the theorem ~~can be~~ proved exactly in the same way.

Case II: Let the forces meet at a point.

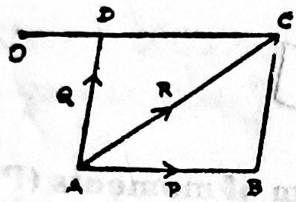


Fig.10(a)

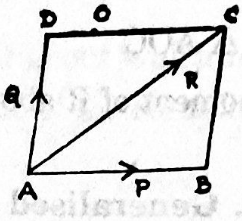


Fig.10(b)

Let the two forces P and Q act at A as shown in figs 10(a) and 10(b) and let O be any point in their plane. Through O, draw a line parallel to the direction of P meeting the line of action of Q at D. Choose the scale of representation such that length AD represents Q in magnitude. On the same scale, let length AB represent P. Complete the paralellogram BAD so that the diagonal AC represents the resultant R of P and Q.

In either figure, the moments of P, Q, R, about O are represented by $2\triangle AOB$, $2\triangle AOD$ and $2\triangle AOC$ respectively.

If fig. 10(a), O lies outside the $\angle$ BAD and the moments of P and Q are both positive.

The algebraic sum of the moments of P and Q

$= 2\,\triangle AOB + 2\triangle AOD$

$= 2\,\triangle ACB + 2\triangle AOD$ ($\because \triangle AOB = \triangle ACB$)

$= 2\,\triangle ADC + 2\triangle AOD$ ($\because$ diagonal AC bisects the ||gm.)

$= 2\,(\triangle ADC + \triangle AOD)$

$= \boxed{2\,\triangle AOC}$

$=$ moment of R about O.

In fig.10(b), O lies inside the angle BAD. The moment of P about O is positive while that of Q is negative.

The algebraic sum of the moments of P and Q

$= 2\,\Delta AOB - 2\Delta AOD$

$= 2\,\Delta ACB - 2\Delta AOD$

$= 2\,\Delta ADC - 2\Delta AOD$

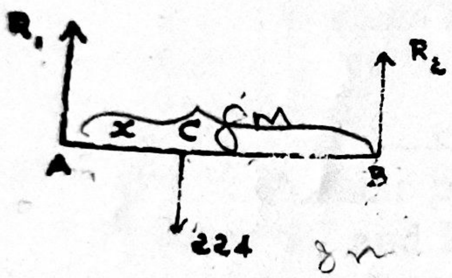
$= 2\,(\Delta ADC - \Delta AOD)$

$= 2\,\Delta\, AOC$

= moment of R about O.

WORKED EXAMPLES

Ex. 4. *Two men carry a load of 224 kg. wt, which hangs from a light pole of length 8 m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other. What is the pressure on each shoulder?*



AB is the light pole of length 8m. C is the point from which the load of 224 kgs. is hung.

Let AC = x. Then BC = 8 − x.

It is given that (8-x) − x = 2.

i.e. 8 − 2x=2 or 2x = 6.

15

Fig. 11

∴ x = 3. i.e. AC = 3 and BC = 5.

Let the pressures at A and B be R_1 and R_2 kg. wt. respectively. Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2 and 224 kg. wt. about any point must be equal to zero.

Taking moments about B,

$224\,CB - R_1.AB = 0$

(as the moment of R_2 about B is 0)

i.e. $224 \times 5 - R_1 \times 8 = 0$.

∴ $R_1 = \frac{224 \times 5}{8} = 140$.

Taking moments about A,

$R_2 AB - 224.AC = 0$.

i.e. $8R_2 - 224 \times 3 = 0$.

$\therefore R_2 = \frac{224 \times 3}{8} = 84$

Note 1. For equilibrium, the weight of 224 kgs must be equal and opposite to the resultant of R_1 and R_2.

$\therefore R_1 + R_2 = 224.$

Hence from this relation, we may find R_2 after finding R_1.

Note 2. In practice, instead of equating the algebraic sum of the moments of the forces about any point to zero, it will be convenient to equate the sum of the moments in one direction to the sum of the moments in the other direction. Hence in the above, taking moments about B, we have $R_1 . AB = 224\ BC$.

HW

Ex. 5 *A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that*

$$\frac{W_1}{W + W_1} + \frac{W_2}{W + W_2} = \frac{b}{a}.$$

16 Let AB be the plank placed upon two vertical props at C and D. CD = b. The weight W of the plank acts at G, the midpoints of AB,

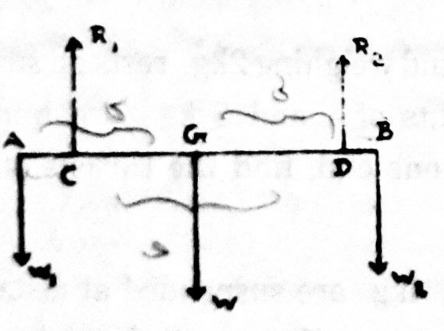


Fig. 12

AG = GB = a

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D then is zero.

There is upward reaction R_1 at C.

Now, taking moments about C, we have

$W_1 . AC = W.CG$

i.e. $W_1\ (AG-CG) = W.CG$

or $W_1.AG = (W + W_1).CG$

i.e. $W_1 a = (W+W_1)CG$

$$\text{or } CG = \frac{W_1 a}{W+W_1} \quad \ldots\ldots\ldots (1)$$

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Now taking moments about D, we get

$W.GD = W_2.BD$

i.e. $W.GD = W_2\,(GB-GD)$

or $GD\,(W+W_2) = W_2.GB = W_2.a$

$$\text{or } GD = \frac{W_2 a}{W+W_2} \quad \ldots\ldots\ldots (2)$$

But $CG + GD = CD = b$

$$\therefore \frac{W_1 a}{W+W_1} + \frac{W_2 a}{W+W_2} = b$$

$$\text{or } \frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}$$