UNIT III

Chapter 4:

Parabolic Differential Equations.

4.1 Occurrence and Derivation of the Diffusion Equation.

The diffusion phenomena such as conduction of heat in solids and diffusion of ~~veloc~~ vorticity in the case of viscous fluid flow past of a body are governed by a partial differential equation of parabolic type.

For example,

The flow of heat in the conducting medium is governed by the parabolic equation $\rho c \frac{\partial T}{\partial t} = \operatorname{div}(k \nabla T) + H(\bar{r}, T, t)$

where ρ is the density, c is the specific heat of the solid, T is the temperature at a point with position vector $\bar{r}$, k is the thermal conductivity, t is the time and $H(\bar{r}, T, t)$ is the amount of heat generated per unit time in the element dV situated at a point (x, y, z)

whose position vector is $\bar{r}$. This equation is known as diffusion equation or heat equation.

To derive the heat equation or Diffusion equation from the basic concepts-

Let V be an arbitrary domain bounded by a closed surface S and let $V^* = V \cup S$. Let $T(x,y,z,t)$ be the temperature at a point $P(x,y,z)$ at time t.

If the temperature is not constant, heat flows from high temperature ~~region~~ region to low temperature region and follows the Fourier law which states that heat flux $\bar{q}(\bar{r},t)$ across the surface element ds with normal $\hat{n}$ is proportional to the gradient of temperature.

$$\therefore \bar{q}(\bar{r},t) = -k \nabla T(\bar{r},t) \quad \text{——— (1)}$$

where k is the thermal conductivity of the body. The negative sign indicates that the heat flux vector points in the direction of decreasing temperature.

Let $\hat{n}$ be the outward unit normal vector and $\bar{q}$ be the heat flux at the surface element ds.

Then the rate of heat flowing out through an element surface ds in unit time is

$$dQ = (\bar{q} \cdot \bar{n})\, ds \quad \text{—— (2)}$$

Heat can be generated due to nuclear reactions or movement of mechanical parts as an ~~unit~~ inertial measurement units (IMU) or due to chemical sources which may be functions of position, temperature and time and may be denoted by $H(\bar{r}, T, t)$.

Also the amount of heat dQ needed to raise the temperature of an element of mass $dm = \rho dV$ to the value T is given by

$$dQ = cT\rho\, dV$$, c being specific heat and ρ is density of substance.

$$\therefore Q = \iiint_V cT\rho\, dV$$

$$\Rightarrow \frac{dQ}{dt} = \iiint_V \rho c \frac{\partial T}{\partial t} dV \quad \text{—— (3)}$$

The energy balance equation for a small control volume is,

"The rate of energy storage in V is equal to the sum of rate of heat entering V through its boundary surface and the rate of heat generation in V.

Thus,

$$\iiint_V \rho c \frac{\partial T(\bar{r},t)}{\partial t} dV = -\iint_S \bar{q}\cdot\hat{n}\, ds + \iiint_V H(\bar{r},T,t)\, dV \quad \text{—— (4)}$$

Using divergence theorem, we get

$$\iint_S \bar{q}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\bar{q})\, dV = \iiint_V \operatorname{div}\bar{q}\, dV$$

∴ (4) becomes

$$\iiint_V \left[\rho c \frac{\partial T(\bar{r},t)}{\partial t} + \operatorname{div}\bar{q}(\bar{r},t) - H(\bar{r},T,t)\right] dV = 0 \quad \text{—— (5)}$$

Since volume is arbitrary, we have

$$\rho c \frac{\partial T(\bar{r},t)}{\partial t} + \operatorname{div}\bar{q}(\bar{r},t) - H(\bar{r},T,t) = 0$$

$$\Rightarrow \rho c \frac{\partial T(\bar{r},t)}{\partial t} = -\operatorname{div}\bar{q}(\bar{r},t) + H(\bar{r},T,t) \quad \text{—— (6)}$$

substituting (1) in (6), we obtain

$$\rho c \frac{\partial T(\bar{r},t)}{\partial t} = \nabla . [K \nabla T(\bar{r},t)] + H(\bar{r},T,t)$$

$$(or)\ \rho c \frac{\partial T(\bar{r},t)}{\partial t} = K \nabla^2 T(\bar{r},t) + H(\bar{r},T,t) \quad \text{—— (7)}$$

$$\frac{\partial T(\bar{r},t)}{\partial t} = \frac{K}{\rho c} \nabla^2 T(\bar{r},t) + \frac{H(\bar{r},T,t)}{\rho c} \quad \text{—— (8)}$$

If we define thermal diffusivity of the medium as $k = \frac{K}{\rho c}$, then the differential equation of heat conduction with heat source is

$$\frac{\partial T(\bar{r},t)}{\partial t} = k \nabla^2 T(\bar{r},t) + \frac{H(\bar{r},T,t)}{\rho c} \quad \text{—— (9)}$$

In the absence of heat sources, Equation (9) reduces to

$$\frac{\partial T(\bar{r},t)}{\partial t} = k \nabla^2 T(\bar{r},t) \quad \text{—— (10)}$$

The fundamental problem of heat conduction is to obtain the solution of equation (9) subject to the initial and boundary conditions which are called Initial Boundary Value Problems (IBVPs).

4.2 Boundary conditions

The boundary conditions are mainly of three types.

Boundary Condition - I

The temperature is prescribed all over the boundary surface. That is the temperature $T(\bar{r}, t)$ is a function of both position and time. This type of boundary condition is called Dirichlet condition.

Sometimes, the temperature on the boundary surface is a function of position only or is a function of time only or a constant.

A special case includes $T(\bar{r}, t) = 0$ on the surface of boundary, which is called a homogeneous boundary condition.

Boundary Condition II

The flux of heat, that is the normal derivative of temperature $\frac{\partial T}{\partial n}$ is prescribed on the surface of the boundary. It may be a function of both position and time.

(ii) $\frac{\partial T}{\partial n} = f(\bar{r}, t)$.

This is called the Neumann Condition. Sometimes, the normal derivatives of temperature may be a function of position only or a function of time only. A special case includes $\frac{\partial T}{\partial n} = 0$ on the boundary. This homogeneous boundary condition is also called insulated boundary condition which states that the heat flow is zero.

Boundary Condition III

A linear combination of the temperature and the heat flux is prescribed on the boundary

(iii) $k\frac{\partial T}{\partial n} + hT = G(\bar{r}, t)$ where k and h are constants.

This type of boundary condition is called Robin's Condition. It means that the boundary surface dissipates heat by convection.

Following Newton's law of cooling, which states that, the rate at which heat is transferred from the body to the surroundings is proportional to the difference in temperature between the body and the surroundings, we have

$$k\frac{\partial T}{\partial n} = h(T - T_a)$$ where T_a is the temperature of surrounding.

Its special case may be taken as

$$k\frac{\partial T}{\partial n} + hT = 0$$ which is homogeneous boundary condition

4.3 Separation of Variables Method

Consider the one dimensional heat conduction equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \quad \text{--- (1)}$$

Let $T(x,t) = X(x)Y(t)$ --- (2)

be the solution of the differential equation (1).

Now $T(x,t) = X(x)Y(t)$

$$\left.\begin{aligned} \frac{\partial T}{\partial t} &= X(x)Y'(t) \\ \frac{\partial T}{\partial x} &= X'(x)Y(t) \\ \frac{\partial T^2}{\partial x^2} &= X''(x)Y(t) \end{aligned}\right\} \text{--- (3)}$$

using (3) in (1), we get

$$X(x)\,Y'(t) = k\,X''(x)\,Y(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{k}\frac{Y'}{Y} = \lambda,$$

λ is a separation constant

Then we have

$$X'' = \lambda X \quad \text{and} \quad Y' = \lambda k Y$$

$$\Rightarrow \frac{d^2X}{dx^2} - \lambda X = 0 \quad \text{—— (4)}$$

and $$\frac{dY}{dt} - \lambda k Y = 0 \quad \text{—— (5)}$$

In solving equations (4) and (5), three distinct cases arise.

Case I. When λ is positive

say $\lambda = \alpha^2$

Then the solution of equations (4) and (5) will have the form

$$X = c_1 e^{\alpha x} + c_2 e^{-\alpha x}, \quad Y = c_3 e^{+k\alpha^2 t} \quad \text{—— (6)}$$

Case II. when λ is negative

say $\lambda = -\alpha^2$

Then the solution of equations (4) and (5) will have the form

$$X = c_1 \cos\alpha x + c_2 \sin\alpha x, \quad Y = c_3 e^{-k\alpha^2 t} \quad \text{—— (7)}$$

Case III when $\lambda = 0$,
Then the solution of equations (4) and (5) will have the form

$$X = c_1 x + c_2 \quad , \quad Y = c_3 \qquad \text{—— (8)}$$

Thus various possible solutions of the one dimensional heat conduction equation (1) are

$$\left.\begin{aligned} T(x,t) &= \left(Ae^{\alpha x} + Be^{-\alpha x}\right) e^{k\alpha^2 t} \\ T(x,t) &= \left(A\cos\alpha x + B\sin\alpha x\right) e^{-k\alpha^2 t} \end{aligned}\right\} \text{— (9)}$$

$$T(x,t) = Ax + B$$

where $A = c_1 c_3$, $B = c_2 c_3$

Example 4.3.1

Show that the solution of the equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ satisfying the conditions

i) $T \to 0$ as $t \to \infty$
ii) $T = 0$ for $x = 0$ and $x = a$ for all $t > 0$
iii) $T = x$ when $t = 0$ and $0 < x < a$

is

$$T(x,t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi}{a} x\right) \exp\left[-\left(\frac{n\pi}{a}\right)^2 t\right]$$

Solution:

Given equation is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)}$$

The possible solutions of (1) are

$T(x,t) = \left(Ae^{\alpha x} + Be^{-\alpha x}\right) e^{\alpha^2 t}$ [Here k=1]

$T(x,t) = \left(A\cos\alpha x + B\sin\alpha x\right) e^{-\alpha^2 t}$

$T(x,t) = Ax + B.$

From the given conditions,

condition (i) demands that $T \to 0$ as $t \to \infty$.

$\therefore$ we reject the first solution.

From condition (ii), the third solution gives us

$0 = A\times 0 + B$ and $0 = A\cdot a + B$

$\Rightarrow A = 0$ and $B = 0$

and ~~here~~ hence $T = 0$ for all t.

This is a trivial solution and hence we also reject the third solution.

Thus the only possible solution satisfying condition (i) is

$T(x,t) = (A\cos\alpha x + B\sin\alpha x)\, e^{-\alpha^2 t}$

Using boundary condition (ii), we have

$0 = (A\cos\alpha 0 + B\sin\alpha 0)e^{-\alpha^2 t}$ and

$0 = (A\cos\alpha a + B\sin\alpha a)e^{-\alpha^2 t}$

$\therefore$ we get $\quad 0 = (A \cdot 1 + B \cdot 0)e^{-\alpha^2 t}$

$\Rightarrow \quad A = 0 \quad$ for all t and

$0 = (A\cos\alpha a + B\sin\alpha a)e^{-\alpha^2 t} \quad (\because A = 0)$

$\Rightarrow \quad \sin\alpha a = 0$

$\Rightarrow \alpha a = n\pi, \quad n \in I$

$\Rightarrow \quad \alpha = \frac{n\pi}{a}, \quad n \in I$

Hence solution is found to be of the form,

$T(x,t) = \left[0 \cdot + B\sin\left(\frac{n\pi}{a}\right)x\right]e^{-\left(\frac{n\pi}{a}\right)^2 t}$

(or) $T(x,t) = B\sin\left(\frac{n\pi}{a}\right)x \, e^{-\left(\frac{n\pi}{a}\right)^2 t}$

Noting that, the heat conduction equation is linear.

Its most general solution is obtained by applying the principle of superposition.

Thus,

$$T(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{a}\right)^2 t} \sin\left(\frac{n\pi}{a}\right)x$$

using condition (iii) we get

$$x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right), \quad 0 < x < a \qquad -(2)$$

which is a half-range Fourier sine series and therefore

$$B_n = \frac{2}{a}\int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx$$

put $\frac{\pi x}{a} = z \Rightarrow x = z\frac{a}{\pi}$

$$\Rightarrow dx = \frac{a}{\pi} dz$$

For $x = 0$, $z = 0$

$x = a$, $z = \pi$

$\therefore$ we have

$$B_n = \frac{2}{a}\int_0^{\pi}\left(z\frac{a}{\pi}\right)(\sin(nz))\frac{a}{\pi}dz$$

$$B_n = \frac{2}{a}\frac{a^2}{\pi^2}\int_0^{\pi} z \sin(nz)\, dz$$

Integrating by parts, we obtain

$$B_n = \frac{2a}{\pi^2}\left[\left[-\frac{z\cos nz}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nz)dz\right]$$

$$= \frac{2a}{\pi^2}\left[-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\left[\sin nz\right]_0^{\pi}\right]$$

$$B_n = -\frac{2a\cos n\pi}{\pi n}$$

$$B_n = -\frac{2a}{\pi}\frac{(-1)^n}{n}, \quad n \in I \qquad -\cancel{(3)}$$

$$B_n = \frac{2a}{\pi}\frac{(-1)^{n-1}}{n}, \quad n \in I \qquad -(3)$$

put ③ in ②, we get the required solution is

$$T(x,t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi}{a}x\right) e^{-\left(\frac{n\pi}{a}\right)^2 t}.$$

Example 4.3 2

The ends A and B of a rod 10 cm in length are kept at temperature 0°c and 100°c ~~sep~~ respectively until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20°c and the end B is decreased to 60°c. Find the temperature distribution in rod at time t.

Solution.

The problem is described by

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 10$$

subject to the conditions

$T(0,t) = 0$, $T(10,t) = 100°$

For steady state $\frac{d^2T}{dx^2} = 0$,

whose solution is $T_s = Ax + B$

Now for $x=0$, $T=0$ implies that $B=0$

$$\therefore T_s = Ax$$

and for $x=10$, $T=100^\circ$ implies that

$$100^\circ = A10 \implies A = 10$$

Thus the initial steady temperature distribution in rod is $T_s(x) = 10x$.

Similarly, when the temperature at the ends A and B are changed to $20^\circ C$ and $60^\circ C$, the final steady temperature in rod is

$$T_s(x) = 4x + 20$$

which will be attained after a long time.

At any instant of time the temperature $T(x,t)$ in rod is given by

$$T(x,t) = T_1(x,t) + T_s(x)$$

where $T_1(x,t)$ is the transient temperature distribution which tends to zero as $t \to \infty$.

Now $T(x,t)$ satisfies the given PDE. Hence its general solution is of the form

$$T(x,t) = 4x+20 + e^{-k\lambda^2 t}(B\cos\lambda x + C\sin\lambda x)$$

Using the boundary condition,

~~T=40 for x=0~~

$T=20$, when $x=0$, we obtain

$$20 = 4\times 0+20+e^{-k\lambda^2 t}(B\cos 0 + C\sin 0)$$

$$20 = 20 + Be^{-k\lambda^2 t}$$

$$\Rightarrow B=0, \quad t>0.$$

Using the boundary condition

$T=60$ when $x=10$, we get

$$60 = 4\times 10 + 20 + e^{-k\lambda^2 t}(0 + C\sin 10\lambda)$$

$$\Rightarrow 60 = 60 + C e^{-k\lambda^2 t}\sin(10\lambda)$$

$$\Rightarrow \sin 10\lambda = 0$$

$$\Rightarrow \lambda = \frac{n\pi}{10}, \quad n\in I$$

The principle of superposition yields

$$T(x,t) = 4x+20+\sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{10}\right)^2 t}\cdot\sin\left(\frac{n\pi}{10}x\right)$$

Using the initial condition $T=10x$ when $t=0$, we obtain

$$10x = 4x+20+\sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{10}\right)^2\times 0}\sin\left(\frac{n\pi}{10}x\right)$$

or $$10x = 4x+20+\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)$$

$$\Rightarrow 6x-20 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)$$

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where

$$C_n = \frac{2}{10}\int_0^{10}(6x-20)\sin\left(\frac{n\pi}{10}x\right)dx$$

$$C_n = \frac{-1}{5}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right].$$

Thus the required solution is

$$T(x,t) = 4x+20 + \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]. \sin\left(\frac{n\pi}{10}x\right)e^{-k\left(\frac{n\pi}{10}\right)^2 t}$$

4.4: Diffusion Equation in Cylindrical Coordinates

To find the Solution of Diffusion Equation in cylindrical coordinates.

Pf Consider a three-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = k\nabla^2 T.$$

or $$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t}$$

In cylindrical coordinates (r, θ, z) it becomes

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \quad (1)$$

where $T(r, \theta, z, t)$

Let us assume separation of variables in the form

$$T(r,\theta,z,t) = R(r)\,\Theta(\theta)\,Z(z)\,\phi(t)$$

From this we get

$$\left.\begin{aligned} \frac{\partial T}{\partial r} &= R'\Theta Z\phi, \quad \frac{\partial^2 T}{\partial r^2} = R''\Theta Z\phi \\ \frac{\partial T}{\partial \theta} &= R\Theta' Z\phi, \quad \frac{\partial^2 T}{\partial \theta^2} = R\Theta'' Z\phi \\ \frac{\partial T}{\partial z} &= R\Theta Z'\phi, \quad \frac{\partial^2 T}{\partial z^2} = R\Theta Z''\phi \\ \frac{\partial T}{\partial t} &= R\Theta Z\phi' \end{aligned}\right\} \quad -(2)$$

Using (2) in (1) we get

$$\frac{1}{k} R\Theta Z\phi' = R''\Theta Z\phi + \frac{1}{r} R'\Theta Z\phi + \frac{1}{r^2} R\Theta'' Z\phi + R\Theta Z''\phi$$

$$\frac{1}{k}\frac{\phi'}{\phi} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z}$$

(ie) $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \frac{1}{k}\frac{\phi'}{\phi} = -\lambda^2$

where $-\lambda^2$ is a separation parameter

Now $\frac{1}{k}\frac{\phi'}{\phi} = -\lambda^2$

$$\Rightarrow \phi' + k\lambda^2\phi = 0 \quad \text{———} (3)$$

and

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2 \text{ (say)}$$

Thus, the equation determining Z, R and Θ becomes

$$Z'' - \mu^2 Z = 0 \quad \text{———} \quad (4)$$

and

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + (\lambda^2 + \mu^2) = -\frac{\Theta''}{\Theta}\cdot\frac{1}{r^2}$$

(or)

$$r^2\left(\frac{R''}{R}\right) + r\frac{R'}{R} + (\lambda^2 + \mu^2)r^2 = -\frac{\Theta''}{\Theta} = \Omega^2$$

$\therefore$

$$-\frac{\Theta''}{\Theta} = \Omega^2$$

$$\Rightarrow \Theta'' + \Omega^2\Theta = 0 \quad \text{———} \quad (5)$$

and

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + (\lambda^2 + \mu^2) - \frac{\Omega^2}{r^2} = 0$$

(or)

$$R'' + \frac{1}{r}R' + \left(\lambda^2 + \mu^2 - \frac{\Omega^2}{r^2}\right)R = 0 \quad \text{———} \quad (6)$$

Equations (3) to (5) have particular solutions of the form

$$\phi = e^{-k\lambda^2 t}$$

$$\Theta = C\cos\Omega\Theta + D\sin\Omega\Theta$$

$$Z = Ae^{\mu z} + Be^{-\mu z}$$

The differential Equation (6) is called Bessel's Equation of order Ω and its general solution is known as

$$R(r) = C_1 J_\Omega\left(\sqrt{(\lambda^2+\mu^2)}\, r\right) + C_2 Y_\Omega\left(\sqrt{(\lambda^2+\mu^2)}\, r\right)$$

where $J_\Omega(r)$ and $Y_\Omega(r)$ are Bessel functions of order Ω of the first and second kind respectively.

Equation (6) is singular for $r=0$, the physically meaningful solution must be twice continuously differentiable in $0 \le r \le a$.

Hence Equation (6) has only one bounded solution

(i) $R(r) = J_\Omega\left[\sqrt{(\lambda^2+\mu^2)}\, r\right)$

Finally the general solution of equation (1) is given as

$$T(r, \theta, z, t) = e^{-k\lambda^2 t}\left[A e^{\mu z} + B e^{-\mu z}\right] \times \left[\left[C \cos \Omega\theta + D \sin \Omega\theta\right] J_\Omega\left[\sqrt{(\lambda^2+\mu^2)}\, r\right]\right]$$

4.5. Diffusion Equation in Spherical coordinates

We shall examine the solution of diffusion or heat conduction equation in the spherical coordinates.

Consider the heat conduction equation $\frac{\partial T}{\partial t} = k \nabla^2 T$.

(or) $\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t}$

In the spherical polar coordinates, the above equation can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} = \frac{1}{k}\frac{\partial T}{\partial t} \quad \text{—} (1)$$

Let the solution of eqn (1) be

$$T = R(r)H(\theta)\phi(\phi)\psi(t) \quad \text{———} (2)$$

From this

$$\left.\begin{aligned} \frac{\partial T}{\partial r} &= R' H \phi \psi, \quad \frac{\partial^2 T}{\partial r^2} = R'' H \phi \psi \\ \frac{\partial T}{\partial \theta} &= R H' \phi \psi, \\ \frac{\partial T}{\partial \phi} &= R H \phi' \psi, \quad \frac{\partial^2 T}{\partial \phi^2} = R H \phi'' \psi \end{aligned}\right\} \text{—} (3)$$

$$\frac{\partial T}{\partial t} = R H \phi \psi'$$

using (3) in (1) we get

$$R''H\Phi\psi + \frac{2}{r}R'H\Phi\psi + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, RH'\Phi\psi\right)$$
$$+ \frac{1}{r^2\sin^2\theta}RH\Phi''\psi = \frac{1}{k}RH\Phi\psi'$$

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2\sin^2\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) +$$
$$\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{1}{k}\frac{\psi'}{\psi} = -\lambda^2$$

where λ^2 is separation parameter.

Thus
$$\frac{1}{k}\frac{1}{\psi}\frac{d\psi}{dt} = -\lambda^2$$

$$\Rightarrow \frac{d\psi}{dt} + \lambda^2 k\psi = 0$$

whose Solution is $\psi = C_1 e^{-k\lambda^2 t}$ —— (4)

Also

$$\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{Hr^2\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \right.$$
$$\left.\lambda^2\right] = -\frac{1}{(r^2\sin^2\theta)\Phi}\frac{d^2\Phi}{d\phi^2}$$

$$\Rightarrow r^2\sin^2\theta\left[\frac{1}{R}\left[R'' + \frac{2}{r}R'\right] + \frac{1}{Hr^2\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)\right.$$
$$\left. + \lambda^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

which gives

$$-\frac{1}{\phi}\frac{d^2\phi}{d\varphi^2} = m^2$$

$$\Rightarrow \frac{d^2\phi}{d\varphi^2} + m^2\phi = 0$$

whose solution is

$$\phi = c_1 e^{im\varphi} + c_2 e^{-im\varphi} \quad \text{——— (5)}$$

Now, the other separated equation is

$$\cancel{\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{R}\right.}$$

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right] + \frac{1}{Hr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \lambda^2 = \frac{m^2}{r^2\sin^2\theta}$$

or $$\frac{r^2}{R}\left[R'' + \frac{2}{r}R'\right] + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)$$

or $$\frac{r^2}{R}\left[R'' + \frac{2}{r}R'\right] + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) = n(n+1)$$ (say)

on rearrangement, this equation reduces to

$$\frac{r^2}{R}\left[R'' + \frac{2}{r}R'\right] + \lambda^2 r^2 = n(n+1)$$ and

$$\frac{m^2}{\sin^2\theta} - \frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) = n(n+1)$$

or $R'' + \frac{2}{R} R' + \lambda^2 r^2 \times \frac{R}{r^2} = [n(n+1)] \times \frac{R}{r^2}$

or $R'' + \frac{2}{R} R' + \left[\lambda^2 - \frac{(n)(n+1)}{r^2}\right] R = 0$ —— (6)

and

~~$\frac{d^2H}{d\theta^2} + \cot\theta$~~

$\frac{1}{H\sin\theta}\left[\cos\theta \frac{dH}{d\theta} + \sin\theta \frac{d^2H}{d\theta^2}\right] +$

$$n(n+1) - \frac{m^2}{\sin^2\theta} = 0$$

or $\frac{d^2H}{d\theta^2} + \cot\theta \frac{dH}{d\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] H = 0$ —— (7)

Let ~~$R = (\lambda r)^{-1/2} \beta(r)$, then equation (6) becomes~~

Let $R = (\lambda r)^{-1/2} \beta(r)$, then equation (6) becomes

$$\beta''(r) + \frac{1}{r}\beta'(r) + \left[\lambda^2 - \frac{(n+1/2)^2}{r^2}\right]\beta(r) = 0$$

This is Bessel's differential equation of order $(n+1/2)$ whose solution is

$$\beta(r) = A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r)$$

$\therefore R(r) = (\lambda r)^{-1/2}\left[A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r)\right]$ —— (8)

where J_n and Y_n are Bessel's functions of first and second kind respectively.

Now put $\cos\theta = \mu$

$\therefore \cot\theta = \mu/\sqrt{1-\mu^2}$

$$\frac{dH}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dH}{d\mu} \qquad [\because \mu = \cos\theta]$$

$$\frac{d^2H}{d\theta^2} = (1-\mu^2)\frac{d^2H}{d\mu^2} - \mu\frac{dH}{d\mu} \quad \text{--- (9)}$$

Using (9) in (7) we get

$$(1-\mu^2)\frac{d^2H}{d\mu^2} - 2\mu\frac{dH}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]H = 0 \quad \text{--- (10)}$$

which is associated Legendre differential equation whose solution is

$$H(\theta) = C\,P_n^m(\mu) + D\,Q_n^m(\mu) \quad \text{--- (11)}$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are associated Legendre functions of degree n and of order m, of first and second kind respectively.

Hence the physically meaningful general solution of diffusion equation in spherical coordinates is of the form

$$T(r,\theta,\phi,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r)\,P_n^m(\cos\theta)\,e^{\pm im\phi - k\lambda^2 t} \quad \text{--- (12)}$$

Here the function $Q_n^m(\mu)$ and $(\lambda r)^{-1/2} Y_{n+1/2}(\lambda r)$ are excluded because these functions have poles at $\mu = \pm 1$, and $r = 0$ respectively.

x———x

Assignment Problem : 4.5.1. to 4.5.4.