UNIT-II

Elliptic Differential Equations.

Section 3.1

Occurrence of the Laplace and Poisson Equations.

In this chapter we shall study various properties and techniques for solving Laplace and Poisson equations which are elliptic in nature.

3.1.1 Derivation of Laplace Equation.

Let two particles of masses m_1 and m_2 be situated at points P and Q at a distance r apart. According to Newton's law of gravitation the magnitude of the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them and is given by

$$F = G\frac{m_1 m_2}{r^2} \quad \text{--- (1)}$$

where G is the gravitational constant.

If $\bar{r} = \overline{PQ}$, the force at Q due to the mass at P is given by

$$\bar{F} = -\frac{m_1 \bar{r}}{r^3}$$

We know that $\nabla(f(r)) = f'(r)\hat{r}$

~~Here~~ Take $f(r) = \frac{1}{r}$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r} = -\frac{\bar{r}}{r^3}$$

$$\therefore \bar{F} = -\frac{m_1 \bar{r}}{r^3} = \nabla\left(\frac{m_1}{r}\right) \quad \text{—— (2)}$$

on assuming unit mass at Q and $G=1$, which is called the intensity of the gravitational force.

Suppose a particle of unit mass moves under the attraction of a particle of mass m_1 at P from infinity to Q, then the work done by the force $\bar{F}$ is

$$\int_{\infty}^{r} \bar{F} \cdot d\bar{r} = \int_{\infty}^{r} \nabla\left(\frac{m_1}{r}\right) \cdot d\bar{r}$$

$$\int_{\infty}^{r} \bar{F} \cdot d\bar{r} = \frac{m_1}{r} \quad \text{—— (3)}$$

This is defined as the potential V at Q due to a particle at P and is denoted by

$$V = -\frac{m_1}{r} \quad \text{———} \quad (4)$$

From Equations (2) and (4) we get

$$\bar{F} = -\nabla V \quad \text{———} \quad (5)$$

Now if we consider a system of particles of masses $m_1, m_2, \ldots m_n$ which are at distances $r_1, r_2 \ldots r_n$ respectively, then the force of attraction per unit mass at Q due to the system is

$$\bar{F} = \sum_{i=1}^{n} \nabla\left(\frac{m_i}{r_i}\right) = \nabla\left[\sum_{i=1}^{n} \frac{m_i}{r_i}\right]$$

$$(\text{or}) \quad \bar{F} = \nabla\left[\sum_{i=1}^{n} \frac{m_i}{r_i}\right] \quad \text{———} \quad (6)$$

The work done by the force acting on a particle is

$$\int_{\infty}^{r} \bar{F} \cdot d\bar{r} = \int_{\infty}^{r} \nabla\left[\sum_{i=1}^{n} \frac{m_i}{r_i}\right] \cdot d\bar{r}$$

$$= \sum_{i=1}^{n} \frac{m_i}{r_i}$$

$$\int_{\infty}^{r} \hat{F} \cdot d\bar{r} = -V \quad \text{———} \quad (7)$$

$$\therefore \nabla^2 V = -\nabla^2 \left[\sum_{i=1}^{n} \frac{m_i}{r_i} \right]$$

$$= -\sum_{i=1}^{n} \nabla^2 \left(\frac{m_i}{r_i} \right), \quad r_i \neq 0$$

~~where~~ $\cancel{\nabla^2 = \frac{\partial^2}{\partial x^2}}$

$$\nabla^2 V = -\sum_{i=1}^{n} \nabla^2 \left(\frac{m_i}{r_i} \right) = 0 \quad \text{——— (8)}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, is called the Laplacian operator.

In the case of continuous distribution of matter of density ρ in a volume $\bar{V}$, we have

$\cancel{V(x,y,z) = \iiint_{\bar{V}} \rho(\xi, \eta, \zeta}$

$$V(x,y,z) = \iiint_{\bar{V}} \frac{\rho(\xi, \eta, \zeta)}{r} \, d\bar{V} \quad \text{——— (9)}$$

wher $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

and Q is outside the body.

It can be verified that

$$\nabla^2 V = 0 \quad \text{——— (10)}$$

which is the Laplace equation.

3.1.2. Derivation of Poisson Equation.

Let S be a close surface consisting of particles of masses $m_1, m_2, \dots m_n$.

Let Q be any point on S and $\sum_{i=1}^{n} m_i = M$, be the total mass inside S.

Let $g_1, g_2 \dots g_n$ be the gravitational fields at Q due to $m_1, m_2, \dots m_n$ respectively, within S.

Also let $\sum_{i=1}^{n} g_i = g$, the entire gravity field at Q.

Then by Gauss' Law, we have

$$\iint_S \bar{g} \cdot d\bar{s} = -4\pi G M \quad \text{where} \quad (1)$$

$M = \iiint_V \rho \, d\bar{v}$, ρ is the mass density function.

Since the gravity field is conservative, we have $\bar{g} = \nabla V$ where V is a scalar potential. (2)

But Gauss divergence theorem, states that

$$\iint_S \bar{g} \cdot d\bar{s} = \iiint_V \nabla \cdot \bar{g} \, dv \quad \text{—— (3)}$$

Also from equation (1), we get

$$\iint_S \bar{g} \cdot d\bar{s} = -4\pi G \iiint_V \rho \, d\bar{v} \quad \text{— (4)}$$

Combining Equations (3) and (4) we have

$$\iint_S \bar{g} \cdot d\bar{s} - \iint_S \bar{g} \cdot d\bar{s} = \iiint_V (\nabla \cdot \bar{g}) \, d\bar{v} + 4\pi G \iiint_V \rho \, d\bar{v}$$

$$0 = \iiint_V \left[(\nabla \cdot \bar{g}) + 4\pi G \rho \right] d\bar{v}$$

$$\Rightarrow \nabla \cdot \bar{g} + 4\pi G \rho = 0$$

$$\Rightarrow \nabla \cdot \bar{g} = -4\pi G \rho$$

Using equation (2), we get

$$\nabla \cdot \nabla V = -4\pi G \rho$$

$$\Rightarrow \nabla^2 V = -4\pi G \rho$$

This equation is known as Poisson's Equation.

3.2 : Boundary Value Proplems.

The function V, in addition to satisfying the Laplace and Poisson equations in bounded Region R in three-dimensional space, should also satisfy certain boundary conditions on the boundary C of this region. Such problems are referred to as Boundary value problems for Laplace and Poisson equations.

Note

1) If a function $f \in C^{(n)}$, then all its derivatives of order n are continuous.

2) If $f \in C^{(0)}$, then we mean that f is continuous.

Boundary value Problems for Laplace equation

There are mainly three types of BVPs for Laplace equation.

I :- First kind of BVP (or) Dirichlet Problem

If $f \in C^{(0)}$, and is prescribed on the boundary C of some finite region R, the problem of determining a function $\phi(x, y, z)$ such that $\nabla^2 \phi = 0$ within R and satisfying $\phi = f$ on C, is called the BVP of first kind or Dirichlet Problem.

II : Second kind of BVP

If $f \in C^{(0)}$ and is prescribed on the boundary C of some finite region R, determining a function $\phi(x, y, z)$ so that $\nabla^2 \phi = 0$ within R while $\frac{\partial \phi}{\partial n}$ is specified at every point of C, where $\frac{\partial \phi}{\partial n}$ is the normal derivative of ϕ. This problem is called the Neumann Problem.

III : Third type of BVP

The third type of BVP is concerned with the determination of the function $\phi(x, y, z)$ such that $\nabla^2 \phi = 0$ within R, while a

boundary condition of the form
$\frac{\partial \phi}{\partial n} + h\phi = f$, where $h \geq 0$ is specified
at every point of the boundary C.
This is called mixed BVP or
Churchill's Problem.

3.3: Separation of Variables Method.

The method of Separation of variables is applicable to a large number of classical linear homogeneous equations. The choice of the co-ordinate system in general depends on the shape of the body.

For example, consider a two-dimensional Laplace equation in cartesian co-ordinates.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

we assume the solution in the form

$$u(x, y) = X(x)\, Y(y) \quad \text{—— (2)}$$

Substituting (2) in (1) we get

$$\nabla^2 u = X''(x)\, Y(y) + X(x)\, Y''(y) = 0$$

(i.e) we get $X''Y + Y''X = 0$

(c) $\frac{X''}{X} = -\frac{Y''}{Y} = k$ where k is a separation constant.

Three cases arise:

Case I. Let $k>0$.

Then $k = p^2$, p is real.

We get

$$\frac{X''}{X} = -\frac{Y''}{Y} = p^2$$

Then $\frac{d^2X}{dx^2} - p^2 X = 0$ and

$\frac{d^2Y}{dy^2} + p^2 Y = 0$, whose solution

is given by

$$X = c_1 e^{px} + c_2 e^{-px} \quad \text{and}$$

$$Y = c_3 \cos py + c_4 \sin py$$

Thus the solution is

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad (3)$$

Case II Let $k = 0$.

Then $\frac{X''}{X} = -\frac{Y''}{Y} = 0$

$\therefore$ we get $\frac{d^2X}{dx^2} = 0$ and $\frac{d^2Y}{dy^2} = 0$

Integrating twice, we get

$$X = c_5 x + c_6 \text{ and } Y = c_7 y + c_8$$

$\therefore$ The solution is

$$u(x,y) = (c_5 x + c_6)(c_7 y + c_8) \quad \text{--- (4)}$$

Case III Let $k < 0$

Then $k = -p^2$

$$\therefore \frac{X''}{X} = -\frac{Y''}{Y} = -p^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + p^2 \cancel{Y} = 0 \text{ and}$$

$\frac{d^2Y}{dy^2} - p^2 Y = 0$, whose solution is given by

$$X = c_9 \cos px + c_{10} \sin px \text{ and}$$

$$Y = c_{11} e^{pxy} + c_{12} e^{-pxy}$$

Hence the solution in this case is

$$u(x,y) = (c_9 \cos px + c_{10} \sin px)(c_{11} e^{py} + c_{12} e^{-py}) \quad \text{--- (5)}$$

In all these three cases, c_i, $i = 1, 2 \ldots 12$ refer to integration constants, which are calculated by using the Boundary conditions.

For example,

Consider the boundary conditions

$u(x, 0) = 0$, $u(x, a) = 0$,

$u(x, y) \to 0$ as $x \to \infty$ where

$x \geq 0$ and $0 \leq y \leq a$.

The appropriate solution for $u(x, y)$ by the methods of separation of variables obtained above in this case is

$$u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right) \quad \text{--- (6)}$$

~~Since (u, x)~~

Since $u(x, y) \to 0$ as $x \to \infty$,

from equation (6), we have

$$c_1 = 0 \quad \forall y$$

$$\therefore u(x,y) = \left[0 + c_2 e^{-px}\right]\left[c_3 \cos py + c_4 \sin py\right]$$

$$u(x,y) = c_2 e^{-px}\left[c_3 \cos py + c_4 \sin py\right] \quad \text{--- (7)}$$

Also $u(x, 0) = 0$

$\therefore$ From (7) we get

$$c_2 e^{-px}\left[c_3 \cos 0 + c_4 \sin 0\right] = 0$$

$$\Rightarrow c_2 c_3 e^{-px} = 0$$

$$\Rightarrow c_3 = 0 \quad \left[\because c_2 \neq 0, e^{-px} \neq 0 \ \forall x\right]$$

$\therefore$ Eqn (7) becomes

$$u(x,y) = Ae^{-px} \sin py \quad \text{where } A = C_2C_4 \quad \text{--- (8)}$$

Now $u(x,a) = 0$

$\therefore$ From (8) we get

$$0 = Ae^{-px} \sin pa$$

$$\Rightarrow \sin pa = 0 \quad [\because A \neq 0]$$

$$\Rightarrow pa = n\pi, \quad n \in I$$

$$\Rightarrow p = \frac{n\pi}{a}, \quad n = 0, \pm 1, \pm 2, \cdots$$

$\therefore$ (8) becomes

$$u(x,y) = Ae^{-\frac{n\pi x}{a}} \sin \frac{n\pi}{a} y, \quad n = 0, \pm 1 \cdots$$

$$u(x,y) = \sum_n A_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right),$$ A_n being new constant.

This is the required solution in this case.

Note

1. Laplace Equation in plane polar coordinates

Polar coordinates r, θ defined by the relation $x = r\cos\theta$, $y = r\sin\theta$.

Laplace Equation in plane polar coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

2. Laplace equation in cylindrical coordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3. Laplace equation in spherical coordinates r, θ, ϕ defined by the relations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, is given by

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

<u>Example 3.3.1</u>

Show that two dimensional Laplace equation $\nabla^2 v = 0$, in the plane polar coordinates r and θ has the solution of the form $(Ar^n + Br^{-n})e^{\pm in\theta}$ where A, B and n are constants. Determine V if it satisfies $\nabla^2 v = 0$ in the region $0 \le r \le a$, $0 \le \theta \le 2\pi$ and

i) V remains finite as $r \to 0$

ii) $V = \sum_n C_n \cos(n\theta)$, on $r = a$

Solution.

Laplace equation in Polar coordinates is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \quad \text{—— (1)}$$

Let $V = R(r)\,\Theta(\theta)$ —— (2) be the solution of the equation (1).

Now $V = R(r)\,\Theta(\theta)$

$$\left.\begin{aligned}
\frac{\partial V}{\partial R} &= \frac{dR}{dr}\Theta(\theta) \\
\frac{\partial^2 V}{\partial R^2} &= \frac{d^2R}{dr^2}\Theta(\theta) \\
\frac{\partial V}{\partial \theta} &= R(r)\frac{d\Theta}{d\theta}
\end{aligned}\right\} \text{—— (3)}$$

$$\frac{\partial^2 V}{\partial \theta^2} = R(r)\frac{d^2\Theta}{d\theta}$$

using (3) in (1) we get

$$\frac{d^2R}{dr^2}\Theta(\theta) + \frac{1}{r}\frac{dR}{dr}\Theta(\theta) + \frac{1}{r^2}R(r)\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{1}{R}\left[r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right] = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = n^2 \text{ (say)}$$

From this we get

$$\frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0 \quad \text{—— (4) and}$$

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = n^2$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0 \quad \text{—— (5)}$$

From eqn (4) we get

$$\Theta = e^{\pm in\theta} \quad \text{—— (6)}$$

Let $R = r^m$ be the solution of equation (5), we get

$$m^2 - n^2 = 0$$
$$\Rightarrow m = \pm n$$
$$\therefore R = Ar^n + Br^{-n}$$

$\therefore$ the solution of (1) becomes

$$V = \sum_n \left(A_n r^n + B_n r^{-n}\right)\left[C_n' \cos n\theta + D_n' \sin n\theta\right]$$

which is the required result.

Again, (i) V remains finite as $r \to 0$
$\Rightarrow B_n = 0$

$$\therefore V = \sum A_n r^n \left[C_n' \cos n\theta + D_n' \sin n\theta\right]$$

(ii) Now $V = \sum_n C_n \cos(n\theta)$, on $r = a$,

we have

$$\sum_n C_n \cos(n\theta) = \sum A_n a^n [C_n' \cos n\theta + D_n' \sin n\theta]$$

$$\Rightarrow D_n = 0 \ \forall n \text{ and } A_n C_n' a^n = C_n$$

$$\therefore A_n C_n' = \frac{C_n}{a^n}$$

Thus $V(r,\theta) = \sum_n C_n \left(\frac{r}{a}\right)^n \cos(n\theta)$, is

the required solution.

3:4 Laplace equation in Cylindrical Coordinates

To find the solution of Laplace equation in cylindrical coordinates

Soln The Laplace equation in cylindrical coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{--- (1)}$$

Let us assume the solution of (1) is of the form

$$u(r,\theta,z) = R(r)\Theta(\theta)Z(z) \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial r} = \frac{dR}{dr}\Theta(\theta)Z(z), \quad \cancel{\frac{\partial^2 u}{\partial r^2} = \partial^2}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{d^2 R}{dr^2}\Theta(\theta)Z(z)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{d\Theta}{d\theta} R(r) Z(z) \\ \frac{\partial^2 u}{\partial \theta^2} &= \frac{d^2\Theta}{d\theta^2} R(r) Z(z) \\ \frac{\partial u}{\partial z} &= \frac{dZ}{dz} R(r)\Theta(\theta) \\ \frac{\partial^2 u}{\partial z^2} &= \frac{d^2 Z}{dz^2} R(r)\Theta(\theta)\end{aligned}\right\} \quad \text{--- (3)}$$

Using (3) in (1) we get

$$\frac{d^2R}{dr^2}\Theta(\theta)Z(z) + \frac{1}{r}\frac{dR}{dr}\Theta(\theta)Z(z) +$$

$$\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}R(r)Z(z) + \frac{d^2Z}{dz^2}R(r)\Theta(\theta) = 0$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dr} + \frac{1}{r^2\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} +$$

$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0$$

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} +$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \quad \text{--- (4)}$$

Let $\frac{1}{Z}\frac{d^2Z}{dz^2} = m^2$

Then $\frac{d^2Z}{dz^2} = m^2 Z$

or $$\frac{d^2 Z}{dz^2} - m^2 Z = 0$$

From this equation we get

$$Z = e^{\pm mz} \qquad \text{(5)}$$

~~Also from eqn (4), we get~~

Equation (4) becomes

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + m^2 = 0$$

$$\Rightarrow \frac{r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + m^2 r^2 = 0 \qquad \text{(6)}$$

Let $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2$

Then $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + n^2 = 0$

$$\Rightarrow \frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0$$

$$\Rightarrow \Theta = e^{\pm in\theta} \qquad \text{(7)}$$

Equation (6) becomes

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] - n^2 + m^2 r^2 = 0$$

$$r^2\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] - n^2 R + m^2 r^2 R = 0$$

$$\Rightarrow \left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] - \frac{n^2}{r^2}R + m^2 R = 0$$

$$\Rightarrow \frac{d^2R}{dr^2} + \frac{1}{R}\frac{dR}{dr} + \left(m^2 - \frac{n^2}{r^2}\right)R = 0$$

which is the Bessel's equation.

Its general solution is given by

$$R(r) = A\,J_n(mr) + B\,Y_n(mr) \quad \text{--- (8)}$$

Here $J_n(mr)$ and $Y_n(mr)$ are the nth order Bessel functions of first and second kind respectively.

Now equation (2) becomes

$$u = R(r)\,e^{\pm mz}\,e^{\pm in\theta} \quad [\text{using } (5), (7) \text{ and } (8)]$$

Now $R(r) = A\,J_n(mr) + B\,Y_n(mr)$, which remains finite as $r \to 0$, implies that $B = 0$

Hence the most general solution of (1) is

~~$u(r,\theta,z) = J_n(\lambda r)$~~

$$u(r,\theta,z) = J_n(mr)\left[c_1 e^{mz} + c_2 e^{-mz}\right]\left[c_3 \cos n\theta + c_4 \sin n\theta\right]$$

Also $z \to 0$, as $z \to \infty$, provides us

$$c_1 = 0$$

$\therefore$ ~~The~~

$\therefore$ The general solution of ① is

$$u(r,\theta,z) = J_n(mr)\, e^{-mz} \left[C_3 \cos n\theta + C_4 \sin n\theta\right]$$

which is the required solution.