Partial Differential Equations.

Introduction :-

Many physical problems in Science and engineering, when formulated mathematically give rise to partial differential equation (PDE). In order to understand the physical behaviour of the mathematical model, it is necessary to have some knowledge about the mathematical character, properties and the solution of the governing PDE.

An equation which involves several independent variables denoted by $x, y, z, t, \ldots$ a dependent function u of these variables and its partial derivatives with respect to the independent variables such as

$F(x, y, z, t, \ldots, u, u_x, u_y, u_z, u_t, \ldots, u_{xx}, u_{yy}, \ldots u_{xy} \ldots) = 0$ is called a partial differential equation.

Definition :-

The order of a partial differential equation is the order of the highest order partial derivative occurring in the equation.

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Unit - I

Second order partial differential equation

Sec 2.1 :-

Origin of second order partial differential equations.

Consider the function

$$z = f(u) + g(v) + w \rightarrow (1)$$

where f and g are functions of u and v respectively and u, v and w are functions of x and y.

Take

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y} \text{ and}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

Differentiating (1) with respect to x, we get,

$$\frac{\partial z}{\partial x} = f'(u)u_x + g'(v)v_x + w_x$$

$$p = f'(u)u_x + g'(v)v_x + w_x \rightarrow (2)$$

Differentiating (1) with respect to y, we get $\frac{\partial z}{\partial y} = f'(u)u_y + g'(v)v_y + w_y$

$$q = f'(u)u_y + g'(v)v_y + w_y \rightarrow (3)$$

Differentiating (2) with respect to x,

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we get

$$\frac{\partial^2 z}{\partial x^2} = f''(u)u_x^2 + f'(u)u_{xx} + g''(v)v_x^2 + g'(v)V_{xx} + \omega_{xx}$$

$$r = f''(u)u_x^2 + f'(u)u_{xx} + g''(v)v_x^2 + g'(v)V_{xx} + \omega_{xx} \rightarrow (4)$$

Differentiating (2) with respect to y, we get

$$\frac{\partial^2 z}{\partial x \partial y} = f''(u)u_x u_y + f'(u)u_{xy} + g''(v)v_x N_y + g'(v)V_{xy} + \omega_{xy}$$

$$s = f''(u)u_x u_y + g''(v)v_x V_y + f'(u)u_{xy} + g'(v)V_{xy} + \omega_{xy} \rightarrow (5)$$

Differentiating (3) with respect to y we get

$$\frac{\partial^2 z}{\partial y^2} = f''(u)u_y^2 + f'(u)u_{yy} + g'(v)v_y^2 + g'(v)V_{yy} + \omega_{yy}$$

$$t = f''(u)u_y^2 + f'(u)u_{yy} + g''(v)V_y^2 + g'(v)V_{yy} + \omega_{yy} \rightarrow (6)$$

Now the above five equations contain four arbitrary quantities g', f', g'' and f''.

Eliminating these quantities, we get

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$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s-w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t-w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \rightarrow (7)$$

which involves only p, q, r, s, t and known functions of x and y.

$\therefore$ It is a partial differential equations of second order.

If we expand (7) in terms of first column, we get

$$Rr + Ss + Tt + Pp + Qq = W \rightarrow (8)$$

where R, S, T, P, Q and W are known functions of x and y.

$\therefore$ $z = f(u) + g(v) + w$ is a solution of the second order linear partial differential equation (8) which is a particular type of equation and contains dependent variable z.

Example 2.1.1 :-

If $u = f(x+iy) + g(x-iy)$ where f and g are arbitrary functions, show that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

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Solution :-

Given $u = f(x+iy) + g(x-iy)$

To find $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$

Now $\frac{\partial u}{\partial x} = f'(x+iy) + g'(x-iy)$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+iy) + g''(x-iy) \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = f'(x+iy)(i) + g'(x-iy)(-i)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x+iy)(i)(i) + g''(x-iy)(-i)(-i)$$

$$\frac{\partial^2 u}{\partial y^2} = -f''(x+iy) - g''(x-iy) \rightarrow (2).$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is the required result.

Example 2.1.2 :-

If f and g are arbitrary functions of their respective arguments, show that $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Provided $\alpha^2 = 1 - \frac{v^2}{c^2}$.

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Solution :-

Given $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y) \rightarrow (1)$

To find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial t^2}$

Now $\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y)$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = f'(x - vt + i\alpha y)(i\alpha) + g'(x - vt - i\alpha y)(-i\alpha)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y)(i\alpha)(i\alpha) + g''(x - vt - i\alpha y)(-i\alpha)(i\alpha)$$

$$= -\alpha^2 f''(x - vt + i\alpha y) - \alpha^2 g''(x - vt - i\alpha y)$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right] \rightarrow (3)$$

$$\frac{\partial u}{\partial t} = f'(x - vt + i\alpha y)(-v) + g'(x - vt - i\alpha y)(-v)$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x - vt + i\alpha y)(-v)(-v) + g''(x - vt - i\alpha y)(-v)(-v)$$

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right] \rightarrow (4)$$

Adding (2) and (3) we get

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(1-\alpha^2)\left[f''(x-vt+i\alpha y)+g''(x-vt-i\alpha y)\right] \quad (7)$$

using (4) we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{(1-\alpha^2)}{v^2}\left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$ where

$$\alpha^2 = 1-\frac{v^2}{c^2}$$

Thus $u = f(x-vt+i\alpha y)+g(x-vt-i\alpha y)$ is a solution of the given partial differential equation.

Section 2.2 :-

Linear Partial differential Equations with constant coefficients.

An equation of the form

$$F(D, D')z = f(x,y) \rightarrow (1)$$

Where $F(D, D')$ is a differential operator of the type

$$F(D, D') = \sum_r \sum_s C_{rs} D^r D'^s \rightarrow (2)$$

in which the coefficients C_{rs} are constants, $D = \partial/\partial x$ and $\partial/\partial y$ is called a linear partial differential equation with constant coefficients.

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Solution :-

Any most general solution of the corresponding homogeneous linear partial differential equation $F(D, D')z = 0 \rightarrow (3)$ is called the complementary function of equation (1).

Similarly, any particular solution of (1) which contains no arbitrary constant or function is called a particular integral of (1).

Thus the general solution of (1) is the sum of complemetary function (CF) and particular integral (PI) of (1).

$$\therefore z = C.F + P.I.$$

Theorem - 2.2.1 :-

If $u_1, u_2, \ldots u_n$ are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then $\sum_{r=1}^{n} c_r u_r$ where c_r's are arbitrary constants is also a solution.

Proof :-

Given $u_1, u_2, \ldots u_n$ are solutions of the PDE $F(D, D')z = 0 \rightarrow (1)$.

To Prove $\sum_{r=1}^{n} c_r u_r$ is also a solution of (1)

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Since $u_1, u_2, \ldots u_n$ are Solution of the PDE $F(D, D')z = 0$ (1) we get

$$F(D, D')u_r = 0 \rightarrow (2) \quad r = 1, 2, \ldots n$$

Now $F(D, D')c_r u_r = c_r F(D, D')u_r$ and

$$F(D, D')\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} F(D, D')u_r$$ for

any set of function u_r.

Therefore,

$$F(D, D')\sum_{r=1}^{n} c_r u_r = \sum_{r=1}^{n} F(D, D')(c_r u_r)$$

$$= \sum_{r=1}^{n} c_r F(D, D')u_r$$

$$F(D, D')\sum_{r=1}^{n} c_r u_r = 0 \quad (\text{by } (2))$$

From (1), we have $F(D, D')z = 0$

$\therefore$ we get $z = \sum_{r=1}^{n} c_r u_r$ satisfies the given equation $F(D, D')z = 0$ and hence it is a Solution of $F(D, D')z = 0$

Hence the result.

Note :-

The operator $F(D, D') = 0$ is classified into two types :

Reducible and irreducible.

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Reducibile :-

The operator $F(D, D')$ is said to be reducibile if it can be factorized into the linear factors of the type $D + aD' + b$ where a and b are constants.

Irreducibile :-

The operator $F(D, D')$ is said to be irreducibile if it is not irreducibile.

Example :-

Reducibile : $F(D, D') = D^2 - D'^2$

$= (D + D')(D - D')$

Irreducibile : $F(D, D') = D^2 + D'^2$

Theorem 2.2.2 :-

If $\alpha_r D + \beta_r D' + \gamma_r$ is factor of $F(D, D')$ and $\phi_r(\xi)$ is as arbitrary function of the ~~Sig~~ single variable ξ then $u_r = e^{\left(\frac{-\gamma_r x}{\alpha_r}\right)} \phi_r(\beta_r x - \alpha_r y)$ for $\alpha_r \neq 0$

is a solution of $F(D, D') z = 0$.

Proof :-

Given equation is $F(D, D') z = 0 \rightarrow (1)$

To Prove

$u_r = e^{\frac{-\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y)$ is a solution of (1). $\rightarrow (2)$

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Differentiating (2) partially with respect to x, we get

$$\frac{\partial}{\partial x} u_r = -\frac{\vartheta_r}{\alpha_r} e^{\frac{-\vartheta_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) + e^{\left(\frac{-\vartheta_r x}{\alpha_r}\right)} \cdot \phi_r'(\beta_r x - \alpha_r y)\beta_r$$

$$D u_r = \frac{-\vartheta_r}{\alpha_r} u_r + \beta_r \cdot e^{\frac{-\vartheta_r x}{\alpha_r}} \phi_r'(\beta_r x - \alpha_r y) \rightarrow (3)$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial}{\partial y} u_r = e^{\left(\frac{-\vartheta_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y)(-\alpha_r)$$

$$D' u_r = -\alpha_r e^{\frac{-\vartheta_r x}{\alpha_r}} \phi_r'(\beta_r x - \alpha_r y) \rightarrow (4)$$

Now From (3),

$$\alpha_r D u_r = -\vartheta_r u_r + \alpha_r \beta_r e^{-\frac{\vartheta_r x}{\alpha_r}} \phi_r'(\beta_r x - \alpha_r y) \rightarrow (5)$$

From (4),

$$\beta_r D'\cancel{k} u_r = -\beta_r \alpha_r e^{-\frac{\vartheta_r x}{\alpha_r}} \phi_r'(\beta_r x - \alpha_r y) \rightarrow (6)$$

Adding (5) and (6) we get

$$\alpha_r D u_r + \beta_r D' u_r = -\vartheta_r u_r$$

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$$\Rightarrow (\alpha_r D + \beta_r D' + \gamma_r) u_r = 0 \longrightarrow (7)$$

Now

$$F(D, D')\, u_r = \prod_{\substack{s=1 \\ s \neq r}}^{n} (\alpha_s D + \beta_s D' + \gamma_s)\,(\alpha_r D + \beta_r D' + \gamma_r)\, u_r \longrightarrow (8)$$

Combining (7) and (8) we get

$$F(D, D')\, u_r = 0$$

Thus $u_r = e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y)$ is a Solution of $F(D, D')z = 0$.

Definition: -

A Partial differential equation $F(D, D')z = f(x, y)$ is said to be reducible if $F(D, D')$ can be written as a product of linear factors in D and D'.

$$F(D, D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r).$$

Definition: -

Equations which are not reducible are called irreducible equations.

Section 2.3: -

Method of Solving linear Partial differential Equation.

2.3.1. Solution of Reducible Equations: -

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Let $F(D, D')z = f(x, y) \rightarrow (1)$ (13)

be a partial differential equation.

Since equation (1) is reducible, we can write

$$F(D, D')z = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r) z \rightarrow (2)$$

If z satisfies $(\alpha_r D + \beta_r D' + \gamma_r)z = 0$, $r = 0, 1, 2, \ldots n$, then it gives complementary function.

Now $(\alpha_r D + \beta_r D' + \gamma_r)z = 0$ is a linear first order PDE.

$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$ is a linear first order partial differential equation

$$\therefore \frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z} \rightarrow (3)$$

From first two members, we get

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$$

$$\Rightarrow \frac{x}{\alpha_r} = \frac{y}{\beta_r} + C_r$$

$$\Rightarrow \beta_r x - \alpha_r y = c_r \rightarrow (4)$$ c_r is Constant.

Also,

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$$\frac{dx}{\alpha_r} = -\frac{dz}{\gamma_r z}$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = \frac{-\gamma_r}{\alpha_r}$$

$$\Rightarrow \frac{dz}{z} = \frac{-\gamma_r}{\alpha_r}\,dx$$

Integrating, on both sides, we get

$$\log z = \frac{-\gamma_r}{\alpha_r}x + \text{constant}$$

$$z = A_r e^{\left(\frac{-\gamma_r}{\alpha_r}\right)x}$$ where A_r is a constant.

$$\therefore\ z = \phi_r(C_r)\, e^{\left(\frac{-\gamma_r}{\alpha_r}\right)x}$$

$$z = \phi_r(\beta_r x - \alpha_r y)\, e^{\left(\frac{-\gamma_r}{\alpha_r}\right)x} \quad [\text{using } (4)]$$

If $\alpha_r \neq 0$, then

$$CF = \sum_{r=1}^{n} \phi_r(\beta_r x - \alpha_r y)\, e^{\left(\frac{-\gamma_r x}{\alpha_r}\right)}$$

ϕ_r is an arbitrary function.

Particular Case :-

If $\alpha_r = 0$, then from (3), we get

$$\beta_r x = \text{constant} = C_r \quad (\text{say}) \quad \to (5)$$

Also from (3), we have

$$\frac{dy}{\beta_r} = \frac{-dz}{\gamma_r z}$$

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$$-\frac{\gamma_r}{\beta_r}\,dy = \frac{dz}{z}$$

Integrating on both sides, we get

$$-\frac{\gamma_r}{\beta_r}\,y + (\text{constant}) = \log z$$

$$\Rightarrow z = \phi_r(c_r)\cdot e^{\left(-\frac{\gamma_r}{\beta_r}y\right)}$$

$$z = \phi_r(\beta_r x)\, e^{\left(-\frac{\gamma_r}{\beta_r}y\right)} \quad \text{[using (5)]}$$

$\therefore$ In this case,

$$CF = \sum_{r=1}^{n} \phi_r(\beta_r x)\, e^{\left(\frac{-\gamma_r y}{\beta_r}\right)}$$

The above two cases are applicabule, when these is no repeated factor of the type $(\alpha_r D + \beta_r D' + \gamma_r)$.

Solution for the case of Repeated factors:

Solution for the case of the partial differential equation

Let the partial differential equation $F(D, D')z = f(x, y)$ has repeated factors.

Suppose that $(\alpha_r D + \beta_r D' + \gamma_r)^2$ is a factor of $F(D, D')$.

Then we have

$$(\alpha_r D + \beta_r D' + \gamma_r)^2 z = 0 \rightarrow (6)$$

(or) $(\alpha_r D + \beta_r D' + \gamma_r) z = z_1$ where

$$(\alpha_r D + \beta_r D' + \gamma_r) z_1 = 0.$$

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$$\therefore z_1 = \phi_r(\beta_r x + \alpha_r y)\, e^{-\frac{\gamma_r x}{\alpha_r}} \quad (16) \rightarrow (7)$$

we have

$$(\alpha_r D + \beta_r D' + \gamma_r) z = z_1$$

$$= \phi_r(\beta_r x - \alpha_r y)\, e^{-\frac{\gamma_r x}{\alpha_r}}$$

$$\Rightarrow (\alpha_r D + \beta_r D') z = e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) - \gamma_r z$$

$$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} = e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) - \gamma_r z$$

~~Taking the first two~~ members, we get

~~dx~~

$$\therefore \frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) - \gamma_r z}$$

Taking the first two members, we get

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$$

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$$\Rightarrow x\beta_r + \alpha_r y = c_r$$, c_r being a constant

Also, taking the first and the third members, we get

$$\frac{dx}{\alpha_r} = \frac{dz}{e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) - \gamma_r z}$$

$$\Rightarrow \frac{dz}{dx} = \frac{e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) - \gamma_r z}{\alpha_r}$$

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$$\frac{dz}{dx} = \frac{-\nu_r z}{\alpha_r} + \frac{1}{\alpha_r} e^{-\left(\frac{\nu_r x}{\alpha_r}\right)} \phi_r(\beta_r x - \alpha_r y) \quad (14)$$

$$\Rightarrow \frac{dz}{dx} + \frac{\nu_r}{\alpha_r} z = \frac{1}{\alpha_r} e^{-\left(\frac{\nu_r x}{\alpha_r}\right)} \phi_r(c_r)$$

This is a linear ordinary differential equation, whose integrating factor is given by $IF = e^{\int \nu_r/\alpha_r \, dx}$

$$IF = e^{\frac{\nu_r}{\alpha_r} x}$$

Therefore, its solution is given by

$$z e^{\frac{\nu_r}{\alpha_r} x} = \int \frac{1}{\alpha_r} e^{\frac{\nu_r}{\alpha_r} x} e^{-\frac{\nu_r x}{\alpha_r}} \phi_r(c_r) dx$$

$$z e^{\frac{\nu_r x}{\alpha_r}} = \frac{x}{\alpha_r} \phi_r(c_r) + \psi_r(c_r)$$

$$\Rightarrow z = e^{-\frac{\nu_r x}{\alpha_r}} \left[x \phi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_r x - \alpha_r y) \right]$$

This procedure can be generalized up to any order of repetition of factors. Adding this to the sum of the other solutions corresponding to the linear factors without repetition, we get the required complementary function.

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2.3.2:-

Solution of irreducible Equations with constant coefficients:-

Let $F(D, D')z = f(x, y) \rightarrow (1)$

be an irreducible linear partial differential equation with constant coefficients.

Let $F(D, D') = F_1(D, D') F_2(D, D')$ where

F_2 is reducible and F_1 is irreducible.

Since $F_2(D, D')$ is reducible, we get the solutions corresponding to linear factors of $F_2(D, D')$ will be of the type

$$e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y) \text{ if } \alpha_r \neq 0$$

and

$$e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x) \text{ if } \alpha_r \neq 0$$

To find the solutions corresponding to the irreducible factor of $F_1(D, D')$.

We suppose that $z = e^{ax+by}$ is a solution of $F_1(D, D')z = 0$.

$$\therefore F_1(D, D') e^{ax+by} = F(a, b) e^{ax+by}$$

must vanish and hence we get the

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Condition $F(a,b)=0$ $\left[\text{since } \alpha^{a} e^{ax+by} \neq 0\right]$.

$$\therefore z = \sum_{r} c_r e^{a_r x + b_r y} \quad (1a)$$ where

$F_1(a_r, b_r) = 0$, $r = 1, 2, \ldots$ is a complementary function corresponding to the irreducibile factors.

The arbitrary constants a_r and b_r can be chosen depending upon the given conditions.

2.3.3: -

Rules for finding complementary function:-

Consider the equation

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

This can be written as

$$(D^2 + a_1 DD' + a_2 D'^2) z = 0 \longrightarrow (1)$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Then the Auxillary equation is

$$m^2 + a_1 m + a_2 = 0 \longrightarrow (2) \quad m = D/D'$$

Let m_1 and m_2 be the roots of (2)

Case-I: -

When $m_1 \neq m_2$

Then the equation (1) can be written as

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$$(D - m_1 D')(D - m_2 D')z = 0 \rightarrow (3)$$

Now the solution of $(D - m_2 D')z = 0$ will also be a solution of eqn (3).

But $(D - m_2 D')z = 0$

$$\Rightarrow p - m_2 q = 0.$$

This is of Lag[illegible] from form and the Auxillary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

Consider the first two members, we get

$$\frac{dx}{1} = \frac{dy}{-m_2}$$

$$\Rightarrow dy + m_2 dx = 0$$

$$\Rightarrow y + m_2 x = c_1$$

Also,

$$\frac{dx}{1} = \frac{dz}{0}$$

$$\Rightarrow dz = 0$$

$$\Rightarrow z = c_2$$

$\therefore$ $z = f_2(y + m_2 x)$ is a solution of $(D - m_2 D')z = 0$ where f_2 is an arbitrary function of its arguments.

Similarly eqn (3) will also be satisfied by the solution of $(D - m_1 D') = 0$ by $z = f_1(y + m_1 x)$ when f_1 is another arbitrary function.

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Hence the complete solution of (1) is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x). \quad (21)$$

Case - II :-

When two roots are equal.

$$m_1 = m_2 = m \text{ (say)}$$

Then equ (1) can be written as

$$(D - mD')^2 z = 0 \rightarrow (4)$$

Let $(D - mD')z = u$.

Equation (4) becomes

$$(D - mD')u = 0$$

Then by case I, its solution is

$$u = f(y + mx).$$

$\therefore (D - mD')z = u$ takes the form

$$(D - mD')z = f(y + mx)$$

$$\Rightarrow P - mq = f(y + mx)$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(y + mx)}.$$

From the first two members, we get

$$\frac{dx}{1} = \frac{dy}{-m}$$

$$\Rightarrow dy + m\,dx = 0$$

$$\Rightarrow y + mx = c_1$$

$$\Rightarrow dz = f(c_1)\,dx$$

$$\Rightarrow z = x f(c_1) + c_2$$

$$\Rightarrow z = x f(y + mx) + c_2$$

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Thus the complete solution of (1) is

$$z = f_1(y+mx) + x f_2(y+mx) \quad (2$$

Note :-

Generalizing the results of case I and case II.

1> If the roots of AE are $m_1, m_2, \ldots$ all distinct, then

$$CF = f_1(y+m_1x) + f_2(y+m_2x) + \ldots$$

where $f_1, f_2, \ldots$ are all arbitrary function.

2> If two roots of AE are equal.

$m_1 = m_2$ then

$$CF = f_1(y+m_1x) + x f_2(y+m_1x) + f_3(y+m_3x) + \ldots$$

where $f_1, f_2, \ldots$ are all arbitrary function.

3> If three roots of AE are equal.

$m_1 = m_2 = m_3$ then

$$CF = f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+mx) + f_4(y+m_4x) + \ldots$$

2.3.4 :-

Rules for finding particular integral.

See book.

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Example : 2.3.1 :-

b) Solve the equation

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}.$$

Solution :-

Given equation can be written as

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$$

$$F(D, D') = D^3 - 2D^2D' - DD'^2 + 2D'^3$$

The auxillary equation is

$$m^3 - 2m^2 - m + 2 = 0 \quad \text{where } m = D/D'$$

$$(m-1)(m+1)(m-2) = 0$$

The roots are $m = 1, m = -1, m = 2$

$$\therefore C_1 = \phi_1(y-x) + \phi_2(y+x) + \phi_3(y+2x) \rightarrow (1)$$

Particular integral,

$$PI = \frac{1}{(D-D')(D+D')(D-2D')} e^{x+y}$$

$$= \frac{1}{(D-D')}\left[\frac{1}{(1+1)(1-2)}\right] e^{x+y}$$

$$PI = \frac{1}{2}\,\frac{-1}{(D-D')} e^{x+y} \rightarrow (2)$$

Let $w = \frac{1}{(D-D')} e^{x+y}$

$$\Rightarrow (D - D')w = e^{x+y}$$

$$\therefore \frac{dx}{1} = \frac{dy}{-1} = \frac{dw}{e^{x+y}}$$

From first two members,

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\Rightarrow dx + dy = 0$$

$$\Rightarrow x + y = c$$

Also $\frac{dx}{1} = \frac{dw}{e^{x+y}}$

$$\Rightarrow \frac{dw}{e^c} = \frac{dx}{1}$$

$$\Rightarrow dw = e^c dx$$

$$\Rightarrow w = e^c x$$

$$w = x e^{x+y} \rightarrow (3)$$

using (3) in (2) we get

$$PI = -\frac{1}{2} x e^{x+y} \rightarrow (4)$$

From (1) and (4) we get

The complete solution is

$$z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + 2x) - \frac{1}{2} x e^{x+y}.$$

Example - 2 & 3 :-

Refer book.

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classification of second order partial differential equation.

Definition :-

A second order partial differential equation which is linear with respect to the second order partial derivatives r, s and t is said to be a quasi-linear PDE of second order.

Example :-

The equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ where $f(x, y, z, p, q)$ need not be linear, is a quasi-linear PDE.

Note :-

The Coefficients R, S, T may be functions of x and y. However, for the sake of simplicity we assume them to be constants.

Definition :-

The equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0 \rightarrow (1)$

is said to be

i) Elliptic if $S^2 - 4RT < 0$

ii) parabolic if $S^2 - 4RT = 0$ and

iii) Hyperbolic if $S^2 - 4RT > 0$ at a point (x_0, y_0).

If this is true at all the points in

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a domain Ω, then the equation (1) is said to be elliptic, parabolic (or) hyperbolic in that domain. (26)

2.4.1 : Canonical Forms :-

Reduce the PDE

$$Rr + Ss + Tt + f(x, y, z; p, q) = 0$$ to a canonical form.

In order to reduce the PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \rightarrow (1)$$

to a canonical form, we apply the transformation $\xi = \xi(x, y)$, $\eta = \eta(x, y)$

$$\rightarrow (2)$$

Here the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

$$J = \xi_x \eta_y - \xi_y \eta_x \neq 0 \rightarrow (3)$$

in the domain Ω where equation (1) holds. Now we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$p = z_\xi \xi_x + z_\eta \eta_x$$

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$$q = z_{\xi} \xi_y + z_{\eta} \eta_y \qquad (27)$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[z_{\xi} \xi_x + z_{\eta} \eta_x\right]$$

$$= z_{\xi\xi} \xi_x \xi_x + z_{\xi\eta} \xi_x \eta_x + z_{\xi} \xi_{xx} +$$

$$z_{\eta\eta} \eta_x \eta_x + z_{\eta\xi} \xi_x \eta_x + z_{\eta} \eta_{xx}$$

$$r = z_{\xi\xi} \xi_x^2 + z_{\eta\eta} \eta_x^2 + z_{\xi} \xi_{xx} + z_{\eta} \eta_{xx}$$

$$+ 2 z_{\xi\eta} \xi_x \eta_x$$

|| ly

$$t = \frac{\partial^2 z}{\partial y^2} = z_{\xi\xi} \xi_y^2 + z_{\eta\eta} \eta_y^2 + z_{\xi} \xi_{yy} +$$

$$z_{\eta} \eta_{yy} + 2 z_{\xi\eta} \xi_y \eta_y$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = z_{\xi\xi} \xi_x \xi_y + z_{\eta\eta} \eta_x \eta_y +$$

$$z_{\xi\eta} \xi_x \eta_y + z_{\xi\eta} \eta_x \xi_y +$$

$$z_{\eta} \eta_x \eta_y + z_{\xi} \xi_x \xi_y$$

$$= z_{\xi\xi} \xi_x \xi_y + z_{\eta\eta} \eta_x \eta_y + z_{\xi\eta} \left[\xi_x \eta_y + \eta_x \xi_y\right] + z_{\eta} \eta_x \eta_y + z_{\xi} \xi_x \xi_y$$

Subtituting there values

p, q, r, s and t in (1) we get

$$R\left[z_{\xi\xi}\,\xi_x^2 + z_{\eta\eta}\,\eta_x^2 + z_\xi\,\xi_{xx} + z_\eta\,\eta_{xx} + 2z_{\xi\eta}(\xi_x\eta_x)\right] + S\left[z_{\xi\xi}\,\xi_x\xi_y + z_{\eta\eta}\,\eta_x\eta_y + z_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + z_\xi\,\xi_{xy} + z_\eta\,\eta_{xy}\right] +$$

(28)

$$T\left[z_{\xi\xi}\,\xi_y^2 + z_{\eta\eta}\,\eta_y^2 + z_\xi\,\xi_{yy} + z_\eta\,\eta_{yy} + 2z_{\xi\eta}\,\xi_y\eta_y\right] + f\left[x, y, z, z_\xi\xi_x + z_\eta\eta_x, z_\xi\xi_y + z_\eta\eta_y\right] = 0.$$

$$\Rightarrow z_{\xi\xi}\left[R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2\right] +$$

$$z_{\xi\xi}\left[2R\xi_x\eta_x + S(\xi_x\eta_y + \xi_y\eta_x) + 2T\xi_y\eta_y\right] +$$

$$z_{\eta\eta}\left[R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2\right] = F(\xi, \eta, z, z_\xi, z_\eta)$$

$$A(\xi_x, \xi_y)\,z_{\xi\xi} + B(\xi_x, \xi_y, \eta_x, \eta_y)\overset{z_{\xi\eta}}{\cancel{z_{\xi\xi}}}$$

$$+ A(\eta_x, \eta_y)\,z_{\eta\eta} = F(\xi, \eta, z, z_\xi, z_\eta) \longrightarrow (4)$$

when,

$$A(\xi_x, \xi_y) = A(u, v) = Ru^2 + Suv + Tv^2$$

$$B(\xi_x, \xi_y, \eta_x, \eta_y) = B(u_1, v_1, u_2, v_2)$$

$$= 2Ru_1u_2 + S(u_1v_2 + u_2v_1) + 2Tv_1v_2$$

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$$A(\eta_x, \eta_y) = R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2$$

(2a)

To find

$$B^2 - 4A(\xi_x, \xi_y)\, A(\eta_x, \eta_y)$$

$$B^2 - 4A(\xi_x, \xi_y)\, A(\eta_x, \eta_y) = [2Ru_1u_2 + S(u_1v_2 + u_2v_1) + 2Tv_1v_2]^2 - 4(R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2)(R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2)$$

$$= 4R^2u_1^2u_2^2 + S^2(u_1v_2 + u_2v_1)^2 + 4T^2v_1^2v_2^2 + 4RSu_1u_2(u_1v_2 + u_2v_1) + 4ST(u_1v_2 + u_2v_1)v_1v_2 + [8RTu_1u_2v_1v_2] - 4[R^2\xi_x^2\eta_x^2 + RS\xi_x^2\eta_x\eta_y + RT\xi_x^2\eta_y^2 + SR\xi_x\xi_y\eta_x^2 + S^2\xi_x\xi_y\eta_x\eta_y + ST\xi_x\xi_y\eta_y^2 + RT\xi_y^2\eta_x^2 + TS\eta_x\eta_y\xi_y^2 + T^2\xi_y^2 + \eta_y^2]$$

$$\Rightarrow B^2 - 4A(\xi_x, \xi_y)\, A(\eta_x, \eta_y) = (S^2 - 4RT)J \rightarrow (5)$$

where J is $\xi_x\eta_y - \xi_y\eta_x$

case - I : -

$$S^2 - 4RT > 0$$

Under the condition $S^2 - 4RT > 0$ the equation $R\lambda^2 + S\lambda + T = 0$ has real and distinct roots.

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Let these roots be λ_1 and λ_2.

Now, choose φ and η such that

$$\varphi_x = \lambda_1 \varphi_y, \quad \eta_x = \lambda_2 \eta_y \rightarrow (6)$$

Now $\varphi_x = \lambda_1 \varphi_y \Rightarrow \varphi_x - \lambda_1 \varphi_y = 0$ being a first order linear PDE, we have

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\varphi}{0}$$

From $\frac{dx}{1} = \frac{d\varphi}{0} \Rightarrow d\varphi = 0$

$$\Rightarrow \varphi = \text{constant}$$

Also, $\frac{dx}{1} = \frac{dy}{-\lambda_1}$

$$\Rightarrow \frac{dy}{dx} = -\lambda_1$$

$$\Rightarrow \frac{dy}{dx} + \lambda_1 (x, y) = 0 \rightarrow (7)$$

$\parallel^{ly}$ we get $\frac{dy}{dx} + \lambda_2 (x, y) = 0 \rightarrow (8)$.

Let the solutions of these equations (7) and (8) be given by

$$f_1 (x, y) = \text{Constant and}$$

$$f_2 (x, y) = \text{constant.}$$

Thus we get $f_1(x, y) = \varphi$ and $f_2 (x, y) = \eta$

$$\rightarrow (9).$$

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Now $A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$ (31)

$$= \xi_y^2\left[R\frac{\xi_x^2}{\xi_y^2} + \frac{S\xi_x\xi_y}{\xi_y^2} + T\right]$$

$$= \xi_y^2\left[R\lambda_1^2 + S\lambda_1 + T\right]$$

Since λ_1 is a root of the equation

$R\lambda^2 + S\lambda + T = 0$, we get

$R\lambda_1^2 + S\lambda_1 + T = 0$

$\therefore A(\xi_x, \xi_y) = \xi_y^2 (c) = 0$

$\|^{ly}$ Since λ_2 is a root of the equation

$R\lambda^2 + S\lambda + T = 0$, we get

$A(\eta_x, \eta_y) = 0$

$\therefore$ Equation (5) is

$B^2 = (S^2 - 4RT) J \neq 0$

Then equation (4) reduces to

$B(\xi_x, \xi_y, \eta_x, \eta_y) z_{\xi\eta} = F(\xi, \eta, z, z_\xi, z_\eta)$

$$z_{\xi\eta} = g(\xi, \eta, z, z_\xi, z_\eta) \rightarrow (9)$$

Which is a required canonical form for the hyperbolic partial differential equation.

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Case - II :-

$$S^2 - 4RT = 0$$

Then the equation $R\lambda^2 + S\lambda + T = 0$ has equal roots $\lambda_1 = \lambda_2 = \lambda$ (say)

we choose $\xi = f_1(x, y)$

$f_1(x, y) =$ constant is a solution of

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

Since $A(\xi_x, \xi_y) = 0$ and $S^2 - 4RT = 0$,

Equation (5) becomes,

$$B^2 = 0 \Rightarrow B = 0$$

However $A(\eta_x, \eta_y) \neq 0$, otherwise η will depend upon ξ

Now using $A(\xi_x, \xi_y) = 0$ and $B = 0$ is equation (4) we get

$$A(\eta_x, \eta_y)\, z_{\eta\eta} = F(\xi, \eta, z, z_\xi, z_\eta)$$

$$\therefore z_{\eta\eta} = g(\xi, \eta, z, z_\xi, z_\eta)$$

Which is the required canonical form for the parabolic partial differential equation.

Case - III :-

$$S^2 - 4RT < 0$$

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In this case the roots of $R\lambda^2 + S\lambda + T = 0$ (33) are imaginary and therefore ξ and η will be complex.

Let $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$, α, β are real.

$$\therefore \alpha = \tfrac{1}{2}(\xi + \eta), \quad \beta = \tfrac{i}{2}(\eta - \xi)$$

Now,

$$z_\xi = z_\alpha \alpha_\xi + z_\beta \beta_\xi$$

$$= z_\alpha \tfrac{1}{2} + z_\beta \left(\tfrac{-i}{2}\right)$$

$$z_\xi = \frac{1}{2} z_\alpha - \frac{i}{2} z_\beta$$

$$z_{\xi\eta} = \frac{1}{2}\left[z_{\alpha\alpha}\alpha_\eta + z_{\alpha\beta}\beta_\eta\right] - \frac{i}{2}\left[z_{\beta\alpha}\alpha_\eta + z_{\beta\beta}\beta_\eta\right]$$

$$= \frac{1}{2}\left[z_{\alpha\alpha}\frac{1}{2} + z_{\alpha\beta}\frac{i}{2}\right] - \frac{i}{2}\left[z_{\beta\alpha}\frac{1}{2} + z_{\beta\beta}\frac{i}{2}\right]$$

$$= \tfrac{1}{4} z_{\alpha\alpha} + \frac{i}{4} z_{\alpha\beta} - \frac{i}{4} z_{\beta\alpha} + \frac{1}{4} z_{\beta\beta}$$

$$z_{\xi\eta} = \frac{1}{4}\left[z_{\alpha\alpha} + z_{\beta\beta}\right]$$

The desired canonical form a

$$z_{\alpha\alpha} + z_{\beta\beta} = \phi(\alpha, \beta, z, z_\alpha, z_\beta)$$

Which is the required canonical

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form for the elliptic PDE.

Example - 2.4.1 : -

Reduce the PDE

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to Canonical form and hence solve it.

Solution : -

Given equation is

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y^2} \frac{\partial z}{\partial y} \longrightarrow (1)$$

Here $R = y^2$, $S = -2xy$, $T = x^2$

Now $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$.

$\therefore$ the given equation (1) is a parabolic equation.

Now $R\lambda^2 + S\lambda + T = 0$

$$\Rightarrow y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

$$\Rightarrow (y\lambda - x)^2 = 0$$

$$\Rightarrow \lambda = x/y$$

$\therefore$ we have $\frac{dy}{dx} + \lambda(x,y) = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow x\,dx + y\,dy = 0$$

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$\Rightarrow x^2+y^2 = $ cos constant

Let $\xi = x^2+y^2$

Since η is to be taken in such a way that it must be independent of ξ, hence we take $\eta = x^2-y^2$

$$\xi_x = 2x,\quad \xi_y = 2y,\quad \eta_x = 2x,\quad \eta_y = -2y$$

Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x$$

$$= z_\xi (2x) + z_\eta (2x)$$

$$z_x = 2x\,(z_\xi + z_\eta) \rightarrow (2)$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y$$

$$= z_\xi (2y) + z_\eta (-2y)$$

$$z_y = 2y\,(z_\xi - z_\eta) \rightarrow (3)$$

$$z_{xx} = 2(z_\xi + z_\eta) + 2x\left(z_{\xi\xi}\xi_x + z_{\xi\eta}\eta_x + z_{\eta\xi}\xi_x + z_{\eta\eta}\eta_x\right)$$

$$= 2(z_\xi + z_\eta) + 2x\left[z_{\xi\xi}(2x) + z_{\xi\eta}(2x) + z_{\eta\xi}(2x) + z_{\eta\eta}(2x)\right]$$

$$z_{xx} = 2(z_\xi + z_\eta) + 4x^2\left(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}\right) \rightarrow (4)$$

Illy

$$z_{yy} = 2(z_\xi - z_\eta) + 4y^2\left(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}\right) \rightarrow (5)$$

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$$z_x = 2x\left(z_\xi + z_\eta\right)$$

$$z_{xy} = 2x\left[z_{\xi\xi}\,\xi_y + z_{\xi\eta}\,\eta_y + z_{\eta\xi}\,\xi_y + z_{\eta\eta}\,\eta_y\right]$$

$$= 2x\left[z_{\xi\xi}(2y) + z_{\xi\eta}(-2y) + z_{\eta\xi}(2y) + z_{\eta\eta}(-2y)\right]$$

$$z_{xy} = 4xy\left[z_{\xi\xi} - z_{\eta\eta}\right] \longrightarrow (6)$$

Substitute (2), (3), (4), (5) & (6) in (1) we get

$$y^2\left[2(z_\xi + z_\eta) + 4x^2(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta})\right] - 2xy\left[4xy(z_{\xi\xi} - z_{\eta\eta})\right] + x^2\left[2(z_\xi - z_\eta) + 4y^2(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta})\right] = \frac{y^2}{x}\left[2x(z_\xi + z_\eta)\right] - \frac{x^2}{y}\left[2y(z_\xi - z_\eta)\right]$$

$$\left.\begin{aligned} &z_{\xi\xi}\left[4x^2y^2 - 8x^2y^2 + 4x^2y^2\right] + \\ &z_{\xi\eta}\left[8x^2y^2 - 8x^2y^2\right] + z_{\eta\eta}\left[4x^2y^2 + 8x^2y^2 + 4x^2y^2\right] \end{aligned}\right\} = 0$$

$$16x^2y^2 z_{\eta\eta} = 0$$

$$\Rightarrow z_{\eta\eta} = 0$$

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

$$\Rightarrow \frac{\partial z}{\partial \eta} = \text{Constant}$$

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$$\frac{\partial z}{\partial \eta} = A$$

$\Rightarrow Z = A\eta + B$ where A and B are arbitrary functions of ξ.

$$Z = \eta A(\xi) + B(\xi)$$

(we have $\xi = x^2 + y^2$ and $\eta = x^2 - y^2$

$$\therefore z = (x^2 - y^2) A(x^2 + y^2) + B(x^2 + y^2)$$

which is the required solution of (1).

Example - 2.4.2 : -

Reduce the equation

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = ny^{2n-1} \frac{\partial z}{\partial y}$$ to

Canonical form and find its general Solution.

Solution : -

Given equation is

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = ny^{2n-1} \frac{\partial z}{\partial y} \rightarrow (1)$$

Here $R = (n-1)^2$, $S = 0$, $T = -y^{2n}$

Now

$$S^2 - 4RT = 0 + 4(n-1)^2 y^{2n}$$
$$= [2(n-1)y^n]^2 > 0$$

$$S^2 - 4RT > 0$$

$\therefore$ The given PDE is hyperbolic.

Now $R\lambda^2 + S\lambda + T = 0$

$$\Rightarrow (n-1)^2 \lambda^2 - y^{2n} = 0$$

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$$\Rightarrow \lambda^2 = \frac{y^{2n}}{(n-1)^2}$$

$$\Rightarrow \lambda = \pm \frac{y^n}{(n-1)}$$

$\therefore$ We have

$$\frac{dy}{dx} \pm \lambda = 0$$

$$\Rightarrow \frac{dy}{dx} \pm \frac{y^n}{(n-1)} = 0$$

$$\Rightarrow (n-1)\,dy \pm y^n\,dx = 0$$

$$\Rightarrow \frac{(n-1)}{y^n}\,dy \pm dx = 0$$

$$\Rightarrow (n-1)\,y^{-n}\,dy \pm dx = 0$$

$$\Rightarrow \pm dx = -(n-1)\,y^{-n}\,dy$$

$$x \pm y^{1-n} = \text{Constant}$$

$$\therefore \xi = x + y^{1-n}, \quad \eta = x - y^{1-n}$$

Now

$\xi_x = 1$	$\eta_x = 1$
$\cancel{\xi_y = (1y^{-}}$	
$\xi_y = (1-n)y^{-n}$	$\eta_y = -(1-n)y^{-n}$

Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x$$

$$z_x = z_\xi + z_\eta \longrightarrow (2)$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y$$

$$= z_\xi (1-n)y^{-n} + z_\eta\left[-(1-n)y^{-n}\right]$$

$$z_y = (1-n)y^{-n}\left[z_\xi - z_\eta\right] \longrightarrow (3)$$

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$$z_{xx} = z_{\xi\xi}\,\xi_x + z_{\xi\eta}\,\eta_x + z_{\eta\xi}\,\xi_x + z_{\eta\eta}\,\eta_x$$
(3a)

$$z_{xx} = z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta} \rightarrow (4)$$

$$z_{yy} = (1-n)(-n)y^{-n-1}(z_\xi - z_\eta) + (1-n)y^{-n}\left[z_{\xi\xi}\,\xi_y + z_{\xi\eta}\,\eta_y - z_{\eta\xi}\,\xi_y - z_{\eta\eta}\,\eta_y\right]$$

$$= -n(1-n)y^{-n-1}(z_\xi - z_\eta) + (1-n)y^{-n}\left[z_{\xi\xi}(1-n)y^{-n} + z_{\xi\eta}(-(1-n)y^{-n}) - z_{\eta\xi}(1-n)y^{-n} - z_{\eta\eta}(-(1-n)y^{-n})\right]$$

$$z_{yy} = -n(1-n)y^{-1-n}(z_\xi - z_\eta) + (1-n)^2 y^{-2n}\left[z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}\right]$$
$\rightarrow (5)$

Substituting (2), (3), (4) and (5) in (1) we get

$$\left.\begin{aligned}&(n-1)^2\left[z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}\right] - \\ &y^{2n}\left[-n(1-n)y^{-1-n}(z_\xi - z_\eta) + \right.\\ &\left.(1-n)^2 y^{2n}(z_\xi - 2z_{\xi\eta} + z_{\eta\eta})\right]\end{aligned}\right\} = $$

$$ny^{2n-1}\left[z_\xi(1-n)y^{-n}(z_\xi - z_\eta)\right]$$

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$$\left.\begin{array}{l} Z_{\xi\xi}\left[(n-1)^2-(n-1)^2\right]+ \\ Z_{\xi\eta}\left[2(n-1)^2+2(1-n)^2\right]+ \\ Z_{\eta\eta}\left[(n-1)^2-(1-n)^2\right] \end{array}\right\} = 0.$$

$$\Rightarrow 4(n-1)^2 Z_{\xi\eta} = 0$$

$$\Rightarrow Z_{\xi\eta} = 0$$

$$\Rightarrow Z = f_1(\xi) + f_2(\eta)$$

$$Z = f_1(x+y^{1-n}) + f_2(x-y^{1-n})$$

Where f_1 and f_2 are arbitrary functions of their respective arguments.

Section - 2.5

Adjoint Operators :-

Let $Lu = \phi \rightarrow (1)$ where L is the differential operator given by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_n(x)$$

One way of introducing the adjoint differential operator L^* associated with L is to form the product vLu and integrate it over the interval of intor

Let $\int_A^B vLu\,dx = [\quad]_A^B + \int_A^B u L^* v\,dx \rightarrow (2)$

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which is obtained after repeated integration by Parts.

Here L^* is the operator adjoint to L, where the functions u and v are completely arbitrary except that Lu and L^*v should exist.

Definition: -

If the operator $L = L^*$, then L is called a self adjoint operator.

Example - 2.5.1: -

If L is the operator

$$R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x\partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z \quad \longrightarrow (1)$$

and M is the adjoint operator defined by

$$Mw = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x\,\partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw \quad \longrightarrow (2)$$

then show that

$$\iint_S (wLz - zMw)\,dx\,dy = \int_C [u\cos(n,y) + v\cos(n,y)]\,ds$$

where C is the closed curve enclosing an area S and

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where ϵ is the closed cu (45)

$$u = RSw\frac{\partial z}{\partial x} - z\frac{\partial}{\partial x}(Rw) - z\frac{\partial}{\partial y}(Sw) + Pzw \rightarrow (3)$$

$$V = Sw\frac{\partial z}{\partial x} + Tw\frac{\partial z}{\partial x} - z\frac{\partial}{\partial y}(Tw) + Qzw \rightarrow (4)$$

If $R_x + \frac{1}{2}S_y = P$, $\frac{1}{2}S_x + T_y = Q$ show that the operator L is self adjoint.

Solution : -

$$wLz - zMw = w\left[R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x\partial y} + T\frac{\partial^2 z}{\partial y^2} + P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + Zz\right] - z\left[\frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x\partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw\right]$$

$$wLz - zMw = \left[wR\frac{\partial^2 z}{\partial x^2} - z\frac{\partial^2}{\partial x^2}(Rw)\right] + \left[Sw\frac{\partial^2 z}{\partial y\partial x} - z\frac{\partial^2(Sw)}{\partial y\partial x}\right] + \left[Tw\frac{\partial^2 z}{\partial y^2} - z\frac{\partial^2(Tw)}{\partial y^2}\right] + \left[wP\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(Pw)\right] + \left[Qw\frac{\partial z}{\partial y} + z\,\partial/\partial y\,(Qw)\right] + Zzw - zZw$$

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$$= \frac{\partial}{\partial x}\left(\omega R \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z \frac{\partial}{\partial x}(R\omega)\right) + \quad (43)$$

$$\frac{\partial}{\partial y}\left(S\omega \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z \frac{\partial}{\partial y}(S\omega)\right) +$$

$$\frac{\partial}{\partial y}\left(T\omega \frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(z \frac{\partial}{\partial y}(T\omega)\right) +$$

$$\frac{\partial}{\partial x}(P\omega z) + \frac{\partial}{\partial y}(Q\omega z).$$

$$\omega L z - z M \omega = \frac{\partial}{\partial x}\left[\omega R \frac{\partial z}{\partial x} - z \frac{\partial}{\partial x}(R\omega) - z \frac{\partial}{\partial y}(S\omega) + P\omega z\right] + \frac{\partial}{\partial y}\left[S\omega \frac{\partial z}{\partial x} + T\omega \frac{\partial z}{\partial y} - z \frac{\partial}{\partial y}(T\omega) + Q\omega z\right]$$

$$\omega L z - z M \omega = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$ where u and v are given by (3) and (4) respectively.

$$\therefore \iint_S (\omega L z - z M \omega)\, dx\, dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy$$

$$= \int_C (lu + mv)\, ds$$

$$\iint_S (\omega L z - z M \omega)\, dx\, dy = \int_C \left(u \cos(n, x) + v \cos(n, y)\right) ds$$

Which is required result.

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Claim :-

(iii)

The operator L is self adjoint

To Prove $L = L^*$

Now,

$$Mw = \frac{\partial^2 (Rw)}{\partial x^2} + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(P\ldots$$

$$- \frac{\partial}{\partial y}(Qw) + Zw$$

$$= Rw_{xx} + 2R_x w_x + R_{xx} w + S w_{xy}$$

$$+ S_y w_x + S_x w_y + w S_{xy} + T w_{yy}$$

$$+ 2 T_y w_y + w T_{yy} - P w_x - w P_x - Q_y w$$

$$- Q w_y + Zw$$

$$Mw = R w_{xx} + (2R_x + S_y) w_x + S w_{xy} +$$

$$T w_{yy} - P w_x - w P_x - Q_y w - Q w_y +$$

$$(2T_y + S_x) w_y + R_{xx} w + w S_{xy} +$$

$$w T_{yy} + Zw \rightarrow (5)$$

Given

$$R_x + \frac{1}{2} S_y = P \qquad \frac{1}{2} S_x + T_y = Q$$

$$\Rightarrow 2R_x + S_y = P \qquad S_x + 2T_y = 2Q$$

Using this in (5) we get

$$Mw = R w_{xx} + 2P w_x + S w_{xy} + T w_{yy} -$$

$$P w_x + w P_x - Q_y w - Q w_y + 2Q w_y +$$

$$R_{xx} w + w S_{xy} + w T_{yy} + Zw$$

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$= R\omega_{xx} + S\omega_{xy} + T\omega_{yy} + P\omega_x + Q\omega_y +$ (45)

$Z\omega + R_{xx}\omega + \omega S_{xy} + \omega T_{yy} - \omega P_x - Q_y \omega$

$$= R\frac{\partial^2 \omega}{\partial x^2} + S\frac{\partial^2 \omega}{\partial x \partial y} + T\frac{\partial^2 \omega}{\partial y^2} + P\frac{\partial \omega}{\partial x} +$$

$$Q\frac{\partial \omega}{\partial y} + Z\omega + \omega\frac{\partial^2 R}{\partial x^2} + \omega\frac{\partial^2 S}{\partial x \partial y} +$$

$$\omega\frac{\partial^2 T}{\partial y^2} - \omega\frac{\partial P}{\partial x} - \omega\frac{\partial Q}{\partial y}$$

$\Rightarrow L = L^*$ and hence the operator is self ~~adjoinit~~ adjoint.

Note :-

The general ~~po~~ procedure for constructing the adjoint of a differential operator in

i) Put all the coefficients inside the derivatives.

ii) Switch the signs of all odd-order derivatives.

Example - 2.5.2 :-

Construct the operator adjoint to the laplace operator given by $L(u) = u_{xx} + u_{yy}$.

Solution :-

$$\text{Here } L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

We know that

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$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z$$

$\therefore$ we get $R=1$, $S=0$, $T=1$, $P=0$, $Q=0$, $Z=0$

$\therefore$ The adjoint operator to L is defined by

$$L^* w = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}Q(w) + Zw$$

$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$L^* w = w_{xx} + w_{yy}$$

$$L^* u = u_{xx} + u_{yy} = L(u)$$

$$\Rightarrow L^* = L$$

Hence the laplace operator is a self adjoint operator.

2.5.1. Riemann's Method :-

Riemann's method is a way of solving linear hyperbolic equations that are stated in canonical form.

We know that any linear hyperbolic PDE can be written in the form.

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$$L(z) = \frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = f(x,y) \rightarrow (1)$$

where a, b and c are function of x and y. we define the adjoint of the differential operator L in (1) by

$$Mw = L^*(w) = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial}{\partial x}(aw) - \frac{\partial}{\partial y}(bw) + cw \rightarrow (2)$$

Now

$$wL(z) - zM(w) = w\left[\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz\right] -$$

$$z\left[\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial}{\partial x}(aw) - \frac{\partial}{\partial y}(bw) + cw\right]$$

$$= w\frac{\partial^2 z}{\partial x \partial y} + wa\frac{\partial z}{\partial x} + wb\frac{\partial z}{\partial y} + wcz -$$

$$z\frac{\partial^2 w}{\partial x \partial y} + z\frac{\partial}{\partial x}(aw) + z\frac{\partial}{\partial y}(bw) - zcw$$

$$wL(z) - zM(w) = w\frac{\partial^2 z}{\partial x \partial y} + wa\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(aw)$$

$$- z\frac{\partial^2 w}{\partial x \partial y} + wb\frac{\partial z}{\partial y} + z\frac{\partial}{\partial y}(bw) \rightarrow (3)$$

Now

$$\left.\begin{aligned}\frac{\partial}{\partial x}(awz) &= wa\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(aw) \text{ and} \\ \frac{\partial}{\partial y}(wbz) &= wb\frac{\partial z}{\partial x} + z\frac{\partial}{\partial y}(bw)\end{aligned}\right\} \rightarrow (4)$$

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Using (4) in (3) we get

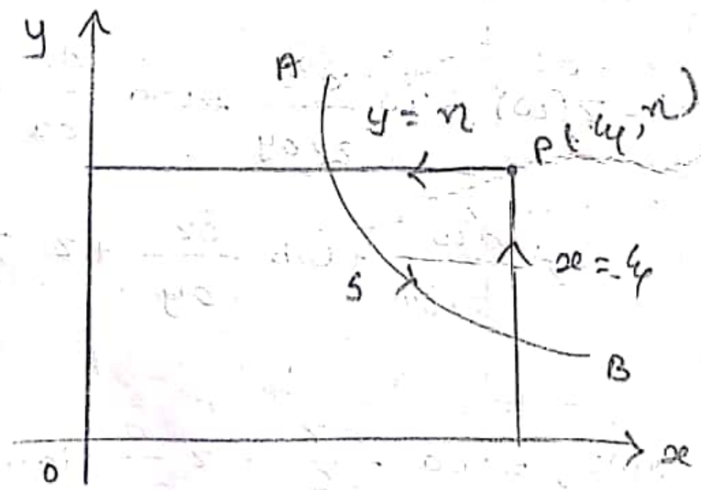
$$wL(z) - zM(w) = w\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(awz) - z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial}{\partial y}(bwz)$$

$$= \frac{\partial}{\partial x}\left[waz - z\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[bwz + w\frac{\partial z}{\partial x}\right]$$

$$wL(z) - zM(w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{where}$$

$$\left.\begin{aligned} u &= waz - z\frac{\partial w}{\partial y} \\ v &= bwz - w\frac{\partial z}{\partial x} \end{aligned}\right\} \rightarrow (5)$$

Now consider an arc AB of a curve Γ where PA is parallel to x-axis PB is parallel to y-axis and $P(\xi, \eta)$ is any point.



Let S denote the area enclosed by the contour ABPA.

Clearly on AP, $y = \eta$, $dy = 0$ and on PB, $x = \xi$, $dx = 0$. (49)

$\therefore$ By Green's theorem

$$\iint_S (\omega L(\omega z) - z M(\omega))\, dx\,dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \int_C u\,dy - v\,dx$$ where C is the closed contour ABPA.

$$\therefore \iint_S (\omega L(z) - z M(\omega))\, dx\,dy = \int_C u\,dy - v\,dx$$

$$= \int_A^B (u\,dy - v\,dx) + \int_B^P (u\,dy - v \times 0) + \int_P^A (u \times 0 - v\,dx)$$

$$\iint_S \omega L z - z M(\omega)\, dx\,dy = \int_A^B (u\,dy - v\,dx) + \int_B^P u\,dy - \int_P^A v\,dx \rightarrow (6)$$

Now

$$\int_P^A v\,dx = \int_P^A \left(\omega b z + \omega \frac{\partial z}{\partial x}\right) dx$$

$$= \int_P^A \left(\omega b z + \omega \frac{\partial z}{\partial x} + z \frac{\partial \omega}{\partial x} - z \frac{\partial \omega}{\partial x}\right) dx$$

$$= \int_P^A \left[\left(\omega b z - z \frac{\partial \omega}{\partial x}\right) + \left[\frac{\partial}{\partial x}(\omega z)\right]\right] dx$$

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$$\therefore \int_P^A v\,dx = \int_P^A \left[\frac{\partial}{\partial x}(\omega z)\right] dx + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx$$

(50)

$$= [\omega z]_P^A + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx$$

$$\int_P^A v\,dx = (\omega z)_A - (\omega z)_P + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx$$

$$\therefore (\omega z)_P = (\omega z)_P + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx - \int_P^A v\,dx \longrightarrow (7)$$

Using (6) in (7) we get

$$[\omega z]_P = (\omega z)_A + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx - \left[\int_B^P u\,dy\right.$$

$$\left.\int_A^B (u\,dy - v\,dx) + \iint_S (\omega L(z) - zM(\omega))\,dx\,dy\right]$$

$$= (\omega z)_A + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx - \int_B^P \left(\omega a z - z\frac{\partial \omega}{\partial y}\right) dy$$

$$- \int_A^B \left(\omega a z - z\frac{\partial \omega}{\partial y}\right) dy - \left(\omega b z + \omega\frac{\partial z}{\partial x}\right) dx +$$

$$\iint_S [\omega L(z) - zM(\omega)]\,dx\,dy$$

$$[\omega z]_P = (\omega z)_A + \int_P^A z\left(\omega b - \frac{\partial \omega}{\partial x}\right) dx -$$

$$\int_B^P z\left(\omega a - \frac{\partial \omega}{\partial y}\right) dy - \int_A^B \omega z\,(a\,dy - b\,dx) +$$

$$\int_A^B \left(z\frac{\partial \omega}{\partial y}dy + \omega\frac{\partial z}{\partial x}dx\right) + \iint_C [\omega L(z) - z\,M(\omega)]\,dx\,dy$$

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This function ω is quite arbitrary, we can choose ω to satisfy the following conditions.

i) $M(\omega)=0$ ii) $\frac{\partial \omega}{\partial y} = a\omega$, on $x=\xi$

iii) $\frac{\partial \omega}{\partial x} = b\omega$, on $y=\eta$. iv) $[\omega]_P = 1$.

Then
$$[z_P] = [\omega z]_A + \int_P^A z(\omega b - b\omega)\,dx - \int_B^P z(\omega a - a\omega)\,dy - \int_A^B \omega z(a\,dy - b\,dx) + \int_A^B \left(z\frac{\partial \omega}{\partial y}dy + \omega\frac{\partial z}{\partial x}dx\right) + \iint_S (\omega f(x,y) - z \times 0)\,dx\,dy$$

$$[z_P] = [\omega z]_A - \int_A^B \omega z(a\,dy - b\,dx) + \int_A^B \left(z\frac{\partial \omega}{\partial y}dy + \omega\frac{\partial z}{\partial x}dx\right) + \iint_S \omega f(x,y)\,dx\,dy \rightarrow (9).$$

This gives the value of z at any point $P(\xi, \eta)$, when the values of z and $\frac{\partial z}{\partial x}$ are given on the curve AB.

However, if the values of x, $\frac{\partial z}{\partial y}$ are given, then

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$$\int_A^B \left[z\frac{\partial \omega}{\partial y}dy + \omega\frac{\partial z}{\partial x}dx\right] = \int_A^B \left[\frac{\partial}{\partial y}(z\omega) - \omega\frac{\partial z}{\partial y}\right]dy$$

$$+\int_A^B \left[\frac{\partial}{\partial x}(\omega z) - z\frac{\partial \omega}{\partial x}\right]dx$$

$$= \int_A^B \left[\frac{\partial}{\partial y}(\omega z)dy + \frac{\partial}{\partial x}(\omega z)dx\right] - \int_A^B \left[\omega\frac{\partial z}{\partial y}dy + z\frac{\partial \omega}{\partial x}dx\right]$$

$$= \left[\omega z\right]_A^B - \int_A^B \left[z\frac{\partial \omega}{\partial x}dx + \omega\frac{\partial z}{\partial y}dy\right]$$

$$\int_A^B \left[z\frac{\partial \omega}{\partial y}dy + \omega\frac{\partial z}{\partial x}dx\right] = \left[\omega z\right]_A^B - \int_A^B \left[z\frac{\partial \omega}{\partial x}dx + \omega\frac{\partial z}{\partial y}dy\right] \rightarrow \cancel{(10)}.$$

$$= \left[\omega z\right]_B - \left[\omega z\right]_A - \int_A^B \left[z\frac{\partial \omega}{\partial x}dx + \omega\frac{\partial z}{\partial y}dy\right] \rightarrow (10)$$

Put (10) in equation (9) we get

$$[z_p] = [\omega z]_A - \int_A^B \omega z(a\,dy - b\,dx) + [\omega z]_B - [\omega z]_A -$$

$$\int_A^B \left[z\frac{\partial \omega}{\partial x}dx + \omega\frac{\partial z}{\partial y}dy\right] + \iint_S \omega f(x,y)\,dx\,dy$$

$$[z_p] = [\omega z]_B - \int_A^B \omega z(a\,dy - b\,dx) -$$

$$\int_A^B \left(z\frac{\partial \omega}{\partial x}dx + \omega\frac{\partial z}{\partial y}dy\right) + \iint_S \omega f(x,y)\,dx\,dy \rightarrow (11)$$

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Adding (9) and (11) we get

$$2[z_P] = [wz]_A + [wz]_B - 2\int_A^B wz(a\,dy - b\,dx) + \int_A^B \left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) - \int_A^B \left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right)$$

$$+ \iint_S w\,f(x,y)\,dx\,dy$$

$$= [wz]_A + [wz]_B - 2\int_A^B wz(a\,dy - b\,dx) + \int_A^B w\left(\frac{\partial z}{\partial x}dx - \frac{\partial z}{\partial y}dy\right) + \int_A^B z\left(\frac{\partial w}{\partial y}dy - \frac{\partial w}{\partial x}dx\right)$$

$$+ \iint_S w f(x,y)\,dx\,dy$$

$$[z_P] = \frac{[wz]_A + [wz]_B}{2} - \int_A^B wz(a\,dy - b\,dx)$$

$$-\frac{1}{2}\int_A^B w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) - \frac{1}{2}\int_A^B z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right)$$

$$+ \iint_S w f(x,y)\,dx\,dy \longrightarrow (12)$$

Equation (11) is used when $z, \frac{\partial z}{\partial y}$ are given on Γ and equation (12) is used when $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are given on Γ.

Example - 2.5.3 :-

Prove that the equation $\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4}z = 0$ the Green's function is given by

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$w(x, y, \xi, \eta) = J_0 \left\{\sqrt{(x-\xi)(y-\eta)}\right\}$ where

$J_0(z)$ is the Bessel function of first kind and of order zero.

Solution! -

The given equation is

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4} z = 0 \rightarrow (1)$$

$L(z) = 0$ where $L = \frac{\partial^2}{\partial x \partial y} + \frac{1}{4}$.

General form is

$$\frac{\partial^2 z}{\partial x \partial y} + a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + cz = f(x, y) \rightarrow (2)$$

where a, b, c are functions of x & y

Here $L = \frac{\partial^2}{\partial x \partial y} + a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c$

Comparing (1) and (2) we get

$a = 0, \; b = 0, \; c = 1/4; \; f(x, y) = 0.$

we know

$$M(w) = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial}{\partial x}(aw) + \frac{\partial}{\partial y}(bw) + cw$$

put $a = 0$, $b = 0$, $c = 1/4$ we get

$$M(w) = \frac{\partial^2 w}{\partial x \partial y} + 0 + 0 + \frac{1}{4} w$$

$$\therefore M = \frac{\partial^2}{\partial x \partial y} + \frac{1}{4}$$

$\Rightarrow M = L$

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$$\therefore\ wL(z) - zM(w) = w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} \quad (55)$$

$$= \frac{\partial}{\partial x}\left(w\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(z\frac{\partial w}{\partial x}\right)$$

$$wL(z) - zM(w) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$$

where $U = w\dfrac{\partial z}{\partial y}$, $V = -z\dfrac{\partial w}{\partial x}$

$$\therefore \iint_S wL(z) - zM(w)\, dx\, dy = \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\, dy$$

$$= \int_{ABPA} (U\,dy - V\,dx)$$

$$= \int_A^B (U\,dy - V\,dx) + \int_B^P U\,dy - \int_P^A V\,dx$$

$$= \int_A^B (U\,dy - V\,dx) + \int_B^P w\frac{\partial z}{\partial y}\,dy + \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$\iint_S \left(wL(z) - zM(w)\right) dx\, dy = \int_A^B (U\,dy - V\,dx) + \int_B^P w\frac{\partial z}{\partial y}\,dy + \int_P^A z\frac{\partial w}{\partial x}\,dx \quad \rightarrow (1)$$

Now $\displaystyle\int_B^P w\frac{\partial z}{\partial y}\,dy$

$$\frac{\partial}{\partial y}(wz) = w\frac{\partial z}{\partial y} + \left(\frac{\partial w}{\partial y}\right) z$$

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$$\Rightarrow w\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(wz) - z\left(\frac{\partial w}{\partial y}\right)$$

$$\therefore \int_B^P w\frac{\partial z}{\partial y}\,dy = \int_B^P\left[\frac{\partial}{\partial y}(wz) - z\left(\frac{\partial w}{\partial y}\right)\right]dy$$

$$\int_B^P w\frac{\partial z}{\partial y}\,dy = [wz]_B^P - \int_B^P z\frac{\partial w}{\partial y}\,dy \rightarrow (2)$$

Using (2) in (1) we get

$$\iint_S [wL(z) - zM(w)]\,dx\,dy = \int_A^B (u\,dy - v\,dx) + [wz]_B^P - \int_B^P z\frac{\partial w}{\partial y}\,dy + \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$= \int_A^B (u\,dy - v\,dx) + [wz]_P - [wz]_B - \int_B^P z\frac{\partial w}{\partial y}\,dy + \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$\therefore [wz]_P = [wz]_B - \int_A^B (u\,dy - v\,dx) + \int_B^P z\frac{\partial w}{\partial y}\,dy - \int_P^A z\frac{\partial w}{\partial x}\,dx + \iint_S [wL(z) - ZH(w)]\,dx\,dy \rightarrow (3)$$

Now suppose that we choose w in such a way that

i) $H(w) = 0$ ii) $\frac{\partial w}{\partial x} = 0$ on $y = \eta$

iii) $\frac{\partial w}{\partial y} = 0$ on $x = \xi$. iv) $[w]_P = 1$.

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Let $w = w(f)$, where f is a single valued differentiable function of x and y. (57)

Let $f^{k} = a(x-\xi)(y-\eta), \; k > 0$

Then $k f^{k-1} \dfrac{\partial f}{\partial x} = a(y-\eta)$ and $\rightarrow (4)$

$$k f^{k-1} \frac{\partial f}{\partial y} = a(x-\xi) \longrightarrow (5)$$

Now $w = w(f)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial f} \frac{\partial f}{\partial x}$$

$$\frac{\partial w}{\partial x} = \frac{a}{k}(y-\eta) f^{1-k} \frac{dw}{df} \quad [\text{using } (5)]$$

To find $\dfrac{\partial^2 w}{\partial x \partial y}$

Now $\dfrac{\partial w}{\partial x} = \dfrac{a}{k}(y-\eta) f^{1-k} \dfrac{dw}{df}$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{a}{k}\left[f^{1-k} \frac{dw}{df} + (y-\eta)(1-k) f^{1-k-1} \frac{\partial f}{\partial y} \frac{dw}{df} + (y-\eta) f^{1-k} \frac{d^2 w}{df^2} \frac{\partial f}{\partial y} \right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{a}{k}\left[f^{1-k} \frac{dw}{df} + (1-k) f^{-k} \frac{a}{k}(x-\xi) f^{1-k} \frac{dw}{df}(y-\eta) + (y-\eta) f^{1-k} \frac{d^2 w}{df^2} \frac{a}{k}(x-\xi) f^{1-k} \right]$$

[using (5)]

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{a}{k}\left[f^{1-k} \frac{dw}{df} + (1-k) \frac{1}{k} f^{-k} f^{k} f^{1-k} \frac{dw}{df} + \frac{1}{k} \frac{d^2 w}{df^2} f^{k} f^{1-k} f^{1-k} \right]$$

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$$\left[\text{using } f^{k} = a(x-\xi)(x-\eta)\right]$$

(58)

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{a}{k}\left[f^{1-k}\frac{d\omega}{df} + (1-k)\frac{1}{k} f^{1-k}\frac{d\omega}{df} + \frac{1}{k}\frac{d^2\omega}{df^2} f^{2-k}\right]$$

$$= \frac{a}{k}\left[\left(1+\frac{1}{k}-1\right) f^{1-k}\frac{d\omega}{df}\right] + \frac{a}{k^2} f^{2-k}\frac{d^2\omega}{df^2}$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{a}{k^2} f^{1-k}\frac{d\omega}{df} + \frac{a}{k^2} f^{2-k}\frac{d^2\omega}{df^2}$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{a}{k^2} f^{2-k} \cdot \frac{d^2\omega}{df^2} + \frac{a}{k^2} f^{1-k}\frac{d\omega}{df} \quad \rightarrow (6)$$

Since $M(\omega) = 0$, we get

$$M(\omega) = \frac{\partial^2 \omega}{\partial x \partial y} + \frac{1}{4}\omega = 0 \quad \rightarrow (7)$$

Subititute (6) in (7), we get

$$\frac{a}{k^2} f^{2-k}\frac{d^2\omega}{df^2} + \frac{a}{k^2} f^{1-k}\frac{d\omega}{df} + \frac{1}{4}\omega = 0$$

$$\frac{a}{k^2} f^{2}\frac{d^2\omega}{df^2} + \frac{a}{k^2} f\frac{d\omega}{df} + \frac{1}{4}\omega f^{k} = 0$$

$$\Rightarrow f^2\frac{d^2\omega}{df^2} + f\frac{d\omega}{df} + \frac{k^2}{4a}\omega f^{4} = 0$$

This becomes a Bessel's equation of order zero if we choose $k=0$ and $a=1$.

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Then its solution is given by

$$w(f) = J_0(f) = J_0\left(\sqrt{(x-\xi)(y-\eta)}\right)$$

We see that conditions (i) to (iv) are satisfied. By putting the value of w in eqn (3) and integrating we can find the solution of the given PDE.

Example - 2.5.4 :-

Verify that the Green's function for the equation $\frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0$

Subject to $z = 0$, $\frac{\partial z}{\partial x} = 3x^2$ on $y = x$ is given by

$$w(x, y, \xi, \eta) = \frac{(x+y)\left[2xy + (\xi-\eta)(x-y) + 2\xi\eta\right]}{(\xi+\eta)^3}$$

and obtain the solution in the form

$$w = (x-y)(2x^2 - xy + 2y^2).$$

Solution :-

Given equation is

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{2}{x+y}\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\text{Here } L(z) = \frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\frac{\partial z}{\partial x} + \frac{2}{x+y}\left(\frac{\partial z}{\partial y}\right) = 0 \quad \rightarrow (1)$$

We know that, the standard canonical form of hyperbolic equation is

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$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = f(x,y) \quad (60)$$
$$\longrightarrow (2)$$

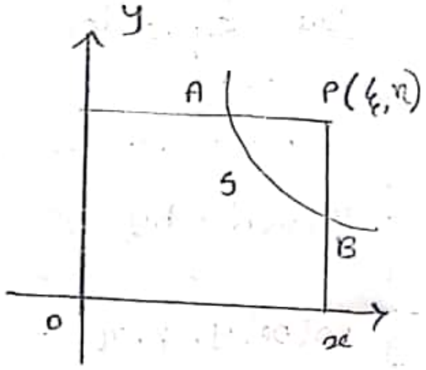
Comparing (1) and (2), we get

$$a = \frac{2}{x+y}, \quad b = \frac{2}{x+y}, \quad c = 0, \quad f = 0.$$

Adjoint equation of (1) is

$M(\omega) = 0$ where

$$M(\omega) = \frac{\partial^2 \omega}{\partial x \partial y} - \frac{\partial}{\partial x}\left(\frac{2\omega}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{2\omega}{x+y}\right)$$



Now suppose that we choose ω in such a way that

i) $M(\omega) = 0$ throughout the xy-plane.

ii) $\dfrac{\partial \omega}{\partial x} = \dfrac{2}{\omega + y}\omega$ on $y = \eta$.

iii) $\dfrac{\partial \omega}{\partial y} = \dfrac{2}{x+y}\omega$ on $x = \xi$.

iv) $[\omega]_P = 1$, at $P(\xi, \eta)$

If we define ω by

$$\omega(x, y, \xi, \eta) = \frac{(x+y)}{(\xi+\eta)^3}\left[2xy + (\xi - \eta)(x - y) + 2\xi\eta\right]$$

$$\frac{\partial \omega}{\partial x} = \frac{1}{(\xi+\eta)^3}\left[(x+y)\left[2y + (\xi - \eta)\right]\right] + \frac{1}{(\xi+\eta)^3}\left[(2xy + (\xi - \eta)(x - y) + 2\xi\eta) \times 1\right]$$

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$$= \frac{1}{(\xi+\eta)^3}\Big[(x+y)\,(2y+\xi-\eta)+2xy+(61)$$
$$(\xi-\eta)(x-y)+2\xi\eta\Big]$$

$$\frac{\partial w}{\partial x} = \frac{1}{(\xi+\eta)^3}\Big[2y(x+y)+(x+y)(\xi-\eta)+$$
$$2xy+(\xi-\eta)(x-y)+2\xi\eta\Big]$$

$$\frac{\partial w}{\partial x} = \frac{1}{(\xi+\eta)^3}\Big[2xy+2y^2+(\xi-\eta)(x+y+x-y)$$
$$+2xy+2\xi\eta\Big]$$

$$\frac{\partial w}{\partial x} = \frac{1}{(\xi+\eta)^3}\Big[4xy+2y^2+2x(\xi-\eta)+2\xi\eta\Big] \rightarrow (3)$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{1}{(\xi+\eta)^3}\Big[4x+4y+0+0\Big]$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{4(x+y)}{(\xi+\eta)^3} \rightarrow (4)$$

$$\frac{\partial w}{\partial y} = \frac{1}{(\xi+\eta)^3}\Big[(x+y)\big[2x-(\xi-\eta)\big]+$$
$$\frac{1}{(\xi+\eta)^3}\Big[\big[2xy+(\xi-\eta)(x-y)\times 1\Big]$$

$$\frac{\partial w}{\partial y} = \frac{1}{(\xi+\eta)^3}\Big[2x(x+y)-(x+y)(\xi-\eta)+$$
$$2xy+(\xi-\eta)(x-y)+2\xi\eta\Big]$$

$$= \frac{1}{(\xi+\eta)^3}\Big[2x^2+2xy+(\xi-\eta)\,[x-y-x-y]$$
$$+2xy+2\xi\eta\Big]$$

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$$\frac{\partial w}{\partial y} = \frac{1}{(\xi+\eta)^3}\left[4xy + 2x^2 - 2y(\xi-\eta) + 2\xi\eta\right] \rightarrow (5)$$

Now

$$H(w) = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial}{\partial x}\left(\frac{2w}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{2w}{x+y}\right)$$

$$H(w) = \frac{\partial^2 w}{\partial x \partial y} - 2\left[\frac{(x+y)\frac{\partial w}{\partial x} - w}{(x+y)^2}\right] - 2\left[\frac{(x+y)\frac{\partial w}{\partial y} - w}{(x+y)^2}\right]$$

$$= \frac{\partial^2 w}{\partial x \partial y} - \frac{2}{(x+y)^2}\left[(x+y)\left[\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right]\right] + \frac{4w}{(x+y)^2}$$

$$H(w) = \frac{\partial^2 w}{\partial x \partial y} - \frac{2}{x+y}\left[\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right] + \frac{4w}{(x+y)^2} \rightarrow (6)$$

Now

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = \frac{1}{(\xi+\eta)^3}\left[8xy + 2(x^2+y^2) + 2(\xi-\eta)(x-y) + 4\xi\eta\right]$$

$$= \frac{1}{(\xi+\eta)^3}\left[2(4xy + x^2 + y^2 + (\xi-\eta)(x-y) + 2\xi\eta)\right]$$

$$= \frac{2}{(\xi+\eta)^3}\left[4xy + x^2 + y^2 + (\xi-\eta)(x-y) + 2\xi\eta\right] \rightarrow (7)$$

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Using (4), (7) and w in (6) we get, (63)

$$M(w) = \frac{4(x+y)}{(\xi+\eta)^3} - \frac{4}{(x+y)^2(\xi+\eta)^3}\left[4xy + x^2 + y^2 + (\xi-\eta)(x-y) + 2\xi\eta\right] +$$

$$\frac{4}{(x+y)^2}\left[\frac{(x+y)\left[2xy + (\xi-\eta)(x-y) + 2\xi\eta\right]}{(\xi+\eta)^3}\right]$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{4}{(x+y)^2(\xi+\eta)^3}\left[4xy + x^2 + y^2\right]$$

$$- \frac{4}{(x+y)^2(\xi+\eta)^3}\left[(\xi-\eta)(x-y) + 2\xi\eta\right] +$$

$$\frac{4}{(x+y)^2}\left[\frac{(x+y)\left(2xy + (\xi-\eta)(x-y) + 2\xi\eta\right)}{(\xi+\eta)^3}\right]$$

$$M(w) = \frac{4(x+y)}{(\xi+\eta)^3} - \frac{4(x+y)^2}{(x+y)(\xi+\eta)^3}$$

$$M(w) = 0$$

$\therefore$ condition (i) is satisfied.

To verify condition (ii)

on $y = \eta$

$$\left(\frac{\partial w}{\partial x}\right)_{y=\eta} = \frac{1}{(\xi+\eta)^3}\left[4x\eta + 2\eta^2 + 2x\xi - 2x\eta + 2\xi\eta\right]$$

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$$= \frac{1}{(\xi+\eta)^3}\left[2x\eta + 2\eta^2 + 2x\xi + 2\xi\eta\right]$$

$$\left(\frac{\partial w}{\partial x}\right)_{y=\eta} = \frac{1}{(\xi+\eta)^3}\left[2\eta^2 + 2x(\xi+\eta) + 2\xi\eta\right] \rightarrow (8)$$

To find $\left(\frac{2w}{x+y}\right)_{y=\eta}$

Now $\frac{2}{x+y}w = \frac{2}{x+y}\left[\frac{x+y}{(\xi+\eta)^3}\left[2xy + (\xi-\eta)(x+y) + 2\xi\eta\right]\right]$

$$\left(\frac{2w}{x+y}\right)_{\eta=y} = \frac{2}{(\xi+\eta)^3}\left[2x\eta + (\xi-\eta)(x-\eta) + 2\xi\eta\right]$$

$$\left(\frac{2}{x+y}w\right)_{y=\eta} = \frac{2}{(\xi-\eta)^3}\left[2x\eta + \xi x - \xi\eta - \eta x + \eta^2 + 2\xi\eta\right]$$

$$= \frac{2}{(\xi-\eta)^3}\left[\eta^2 + 2x\eta - \eta x + \xi x + \xi\eta\right]$$

$$\left(\frac{2w}{x+y}\right)_{y=\eta} = \frac{2}{(\xi+\eta)^3}\left[\eta^2 + x\eta + \xi x + \xi\eta\right]$$

$$\left(\frac{2w}{x+y}\right)_{y=\eta} = \frac{1}{(\xi+\eta)^3}\left[2\eta^2 + 2x(x\xi+\eta) + 2\xi\eta\right] \rightarrow (9)$$

From (8) and (9) we get

$$\left(\frac{\partial w}{\partial x}\right)_{y=\eta} = \frac{2w}{x+y} \text{ at } y=\eta$$

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$\therefore$ condition (ii) is verified.

$\parallel^{ly}$ condition (iii) is verified.

To verify condition (iv)

Now at $x = \xi$, $y = \eta$, we have

$$\omega = \frac{(\xi+\eta)}{(\xi+\eta)^3}\left[2\xi\eta + (\xi-\eta)(\xi-\eta) + 2\xi\eta\right]$$

$$\omega = \frac{(\xi+\eta)}{(\xi+\eta)^3}\left[4\xi\eta + (\xi-\eta)^2\right]$$

$$\omega = \frac{(\xi+\eta)^3}{(\xi+\eta)^3} = 1$$

$$[\omega]_P = 1$$

Hence condition (iv) is also satisfied.

Now

$$\omega L(z) - zH(\omega) = \left[\omega\frac{\partial^2 z}{\partial x\partial y} + \frac{2\omega}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{2\omega}{x+y}\left(\frac{\partial z}{\partial y}\right)\right]$$

$$- \left[z\frac{\partial^2 \omega}{\partial x\partial y} + z\frac{\partial}{\partial x}\left(\frac{2\omega}{x+y}\right) + z\frac{\partial}{\partial y}\left(\frac{2\omega}{x+y}\right)\right]$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \quad \text{where}$$

$$M = \frac{2z\omega}{x+y} - z\frac{\partial \omega}{\partial y} \quad \text{and}$$

$$N = \frac{2z\omega}{x+y} + \omega\frac{\partial z}{\partial x}$$

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$$wL(z) - zM(w) = w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial}{\partial x}\left[\frac{2wz}{x+y}\right]$$
$$+ \frac{\partial}{\partial y}\left[\frac{2wz}{x+y}\right]$$

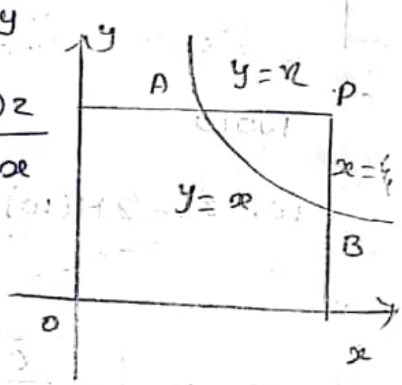
$$= \frac{\partial}{\partial y}\left[w\frac{\partial z}{\partial x}\right] - \frac{\partial}{\partial x}\left[z\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial x}\left[\frac{2wz}{x+y}\right]$$
$$+ \frac{\partial}{\partial y}\left[\frac{2wz}{x+y}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2wz}{x+y} - z\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[\frac{2wz}{x+y} + w\frac{\partial z}{\partial x}\right]$$

$$wL(z) - zM(w) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

where $M = \dfrac{2zw}{x+y} - z\dfrac{\partial w}{\partial y}$ and

$$N = \frac{2zw}{x+y} + w\frac{\partial z}{\partial x}$$



Now using Green's theorem we get,

$$\iint_S (wL(z) - zM(w))\,dx\,dy = \int_C (M\,dy - N\,dx)$$

$$= \int_B^A (M\,dy - N\,dx) + \int_A^P (M\,dy - N\,dx) + \int_P^B (M\,dy - N\,dx) \quad \rightarrow (10)$$

Which on using condition (i) to (iv) and the fact that $y = \eta$ on AP and $x = \xi$ on BP

$y = x \Rightarrow dy = 0$ on AP

$dx = 0$ on BP

Equation (10) becomes

$$LHS = \int_B^A \left[\frac{2zw}{x+y} - z\frac{\partial w}{\partial y}\right] dy - \left[\frac{2zw}{x+y} + w\frac{\partial z}{\partial x}\right] dx$$

$$- \int_A^P \left(\frac{2zw}{x+y} + w\frac{\partial z}{\partial x}\right) dx + \int_P^B \left[\frac{\partial zw}{x+y} - z\frac{\partial w}{\partial y}\right] dy \rightarrow (11).$$

Now,

$$\int_A^P \left(\frac{2zw}{x+y} + w\frac{\partial z}{\partial x}\right) dx = \int_A^P \frac{2zw}{x+y} dx + (wz)_A^P - \int_A^P z\frac{\partial w}{\partial x} dx$$

(11) becomes

$$LHS = \int_B^A \left(\frac{2zw}{x+y} - z\frac{\partial w}{\partial y}\right) dy - \int_B^A \left(\frac{2zw}{x+y} + w\frac{\partial z}{\partial x}\right) dx$$

$$- \int_A^P \frac{2zw}{x+y} dx - (wz)_P + (wz)_A + \int_A^P z\frac{\partial w}{\partial x} dx$$

$$+ \int_P^B \frac{\partial zw}{x+y} dy - \int_P^B z \cdot \frac{\partial w}{\partial y} dy$$

Using condition (ii) to (iv) and

$z = 0$ on $y = x$, we get

$$[z]_P = [zw]_A - \int_B^A w\frac{\partial z}{\partial x} dx$$

Now using the given condition

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$\frac{\partial z}{\partial x} = 9x^2$ on AB, we get

$$[z]_P = [zw]_A - \int_B^A \frac{(3x)^2\left[2x(2x^2+2\xi\eta)\right]}{(\xi+\eta)^3}\,dx$$

$$= \frac{-12}{(\xi+\eta)^3}\int_\xi^\eta (x^5 + x^3\xi\eta)\,dx$$

$$= \frac{-12}{(\xi+\eta)^3}\left[\frac{x^6}{6} + \xi\eta\frac{x^4}{4}\right]_\xi^\eta$$

$$= \frac{-12}{(\xi+\eta)^3}\left[\frac{1}{6}(\eta^6-\xi^6) + \frac{\xi\eta}{4}(\eta^4-\xi^4)\right]$$

$$[z]_P = \frac{-1}{(\xi+\eta)^3}\left[2(\eta^6-\xi^6) + 3\xi\eta(\eta^4-\xi^4)\right]$$

$$= \frac{-1}{(\xi+\eta)^3}\left[2\left[(\eta^2)^3-(\xi^2)^3\right] + 3\xi\eta\left[(\eta^2)^2-(\xi^2)^2\right]\right]$$

$$[z]_P = \frac{\xi^2-\eta^2}{(\xi+\eta)^3}\left[2(\xi^4+\xi^2\eta^2+\eta^4) + 3\xi\eta(\xi^2+\eta^2)\right]$$

$$= (\xi-\eta)(2\xi^2-\xi\eta+2\eta^2)$$

$$\therefore\ z(x,y) = (x-y)(2x^2-xy+2y^2).$$

Hence the result.

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