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Theory of Space Curves

1.1 INTRODUCTION

Differential Geometry is the study of properties of space curves and surfaces with the help of vector calculus. This geometry examines in more details the curves in space and surfaces, whereas the differential geometry of the plane curves deals with the tangents, normals, curvature, asymptotes, involutes, evolutes etc. which have analogues for space curves. Though we have these analogues for a space curve, the curvature at a point of a space curve and the tangent plane at a point to a surface play a dominant rôle in the differential geometry. Thus the differential geometry studies the pointwise properties of space curves and surfaces as distinct from the algebraic geometry whose sole aim is to describe the properties of the configuration as a whole.

In this chapter on space curves, first we shall specify a space curve as the intersection of two surfaces. Then we shall explain how we shall arrive at a unique parametric representation of a point on the space curve and also give a precise definition of a space curve in E_3 as set of points associated with an equivalence class of regular parametric representations. With the help of this parametric representation, we shall define tangent, normal and binormal at a point leading to the moving triad (**t**, **n**, **b**) and their associated tangent, normal and rectifying planes. Since the triad (**t**, **n**, **b**) at a point is moving continuously as P varies over the curve, we are interested to know the arc-rate of rotation of **t**, **n**, and **b**. This leads to the well-known formulae of Serret-Frenet.

Then we shall establish the conditions for the contact of curves and surfaces leading to the definitions of osculating circle and osculating sphere at a point on the space curve and also the evolutes and involutes. Before concluding this chapter, we shall explain what is meant by intrinsic equations of space curves and establish the fundamental theorem of space curves which states that if curvature and torsion are the given continuous functions of a real variable s, then they determine the space curve uniquely. We shall illustrate all the notions developed with a particular type of space curves known as helices. In all our discussion, the basic formula of Serret-Frenet occupies the central position.

1.2 REPRESENTATION OF SPACE CURVES

Whenever we use the word space, it means the Euclidean space of dimension three denoted by E_3. In this space a single equation generally represents a surface and so we need two equations to specify a curve. Thus we first introduce a space curve as the intersection of two surfaces

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0 \qquad \text{...(1)}$$

Though we are able to fix the curve in space with the help of the equations (1), we are unable to obtain the representation of different points on the whole curve. Since a space curve is a set of points with a sense of description, it is desirable to know the coordinates of a point on the curve as functions of single parameter. So the question naturally arises whether one can obtain the parametric representation of a curve with the help of equation (1). First let us examine this question and then its converse.

Let the functions F_1 and F_2 of (1) satisfy the following conditions of the implicit function theorem.

(i) Let F_1 and F_2 have continuous partial derivatives of first order.

(ii) At least one of the following three Jacobian determinants

$$\frac{\partial(F_1, F_2)}{\partial(y, z)}, \quad \frac{\partial(F_1, F_2)}{\partial(z, x)}, \quad \frac{\partial(F_1, F_2)}{\partial(x, y)} \qquad \text{...(2)}$$

is different from zero at a point (x_0, y_0, z_0) on the curve.

Under these conditions one can obtain the solution of two variables in terms of the third variable. Hence without loss of generality, let us assume that the first Jacobian is different from zero. Then we can solve for y and z in terms of x and obtain $y = f_1(x)$ and $z = f_2(x)$ as their solutions. Having obtained these values of y and z in terms of x, we take the parametric representation as

$$x = u, \quad y = f_1(u), \quad z = f_2(u) \qquad \text{...(3)}$$

The above equations treating x as a parameter are true for the restricted values of x under the conditions (2). So they cannot give the parametric representation of the whole curve. Thus having started with the definition of a curve as the intersection of two surfaces, we are unable to arrive at a satisfactory representation of the whole curve.

Next let us examine the converse question in a little more detailed manner. Let us assume that the curve is given in the following parametric form

$$x = x(u), \quad y = y(u), \quad z = z(u), \quad u_1 \le u \le u_2 \qquad \text{...(4)}$$

and find the curve as the intersection of two surfaces. Now solving for u in terms of x from the first equation as $u = f(x)$ and substituting for u in the other two equations, we obtain

$$y = y[f(x)], \quad z = z[f(x)] \qquad \text{...(5)}$$

Hence the equations of two surfaces specifying the curve are given by (5).

The difficulty with this method of elimination is that the intersection of the above two surfaces not only contain the given curve, but also it may contain some extra curves as shown by the following example.

Example 1. Consider the cubic curve with parametric representation

$$x = u,\ y = u^2,\quad z = u^3 \qquad \ldots(1)$$

A most straight forward method of elimination gives

$$y = x^2 \text{ and } xz = y^2 \qquad \ldots(2)$$

We know that $y^2 = xz$ represents a parabolic cylinder and $x^2 = y$ is a cone with the vertex at the origin. Hence the two surfaces specify the curve and their intersection not only contain the cubic curve (1) but also the z-axis as the equations (2) satisfy $x = 0$, $y = 0$.

Another method of elimination of the parameter u in (1) gives $xy = z$ and $xz = y^2$ which are the equations of the two required surfaces specifying the cubic curve. Though $y^2 = xz$ represents the same parabolic cylinder as in (2), $xy = z$ represents a paraboloid. Their intersection contains not only the given cubic curve but also the x-axis as the two equations satisfy the conditions $y = 0$, $z = 0$.

Thus we see that different types of eliminations of the parameters in the parametric equations not only give the curve as the intesection of the two surfaces but also include extraneous curves like z-axis or x-axis. As a result, we are unable to arrive at a one to one correspondence between the points on the curves and the curves given by the intersections of two surfaces whose equations we obtain from the parametric representations.

So, of the two methods of defining a space curve, we find that the definition of a curve as the intersection of the two surfaces gives very little information about the curves. Though it fixes the curve in space, it fails to give the coordinate representation of different points and the sense of description of the curve. So it is not amenable to the treatment of vector calculus.

Summarising the above discussion, we conclude that when the curve is defined as the intersection of two surfaces, the parametric representation obtained from the two surface equations does not give the whole curve and when the curve is represented parametrically, the equations we obtain from the elimination of parameters lead two surface equations giving not only the curve in question but also some extraneous curves. Hence the two definitions are not equivalent.

So between the two definitions of a space curve, we choose the parametric representation which is most advantageous. Since the parametric representation gives the position of different points on the curve and also the sense of variation along the curve, we can represent the points on the curve vectorially and use vector analysis for studying the properties of space curves.

1.3 UNIQUE PARAMETRIC REPRESENTATION OF A SPACE CURVE

The greatest advantage of parametric representation is that it gives the sense of description of the curve as parameter varies in a given interval. However it should be noted that we have different methods of parametrising the same curve. The problem is how we are going to deal with the different parameters representing the same point on the curve. We collect together different parameters representing the same point into a class and choose a representative parameter from the class which yields the same property as the other parameters of the class. Hence a curve will therefore be specified by all possible parametric representations which are equivalent in that all describe the same curve in the same sense. We shall make these notions precise in the following definitions.

Definition 1. Let I be a real interval and m be a positive integer. A real valued function f defined on I is said to be of class m, if f has continuous mth derivative at every point of I. We call such functions C^m functions.

***Note* 1.** When a function is infinitely differentiable, then f is said to be of class infinity or the function itself is called C^∞ functions.

***Note* 2.** If f is a real valued function of several variables, then it is of class m if it admits all continuous partial derivatives of order m.

Definition 2. A function f is said to be analytic over I, if f is single valued and f possesses continuous derivatives of all orders at every point of I. This class of functions is denoted by ω. The function itself is called a C^ω function. These functions have power series representations in the neighbourhood of every point of I.

Definition 3. A vector valued function $\mathbf{R} = \mathbf{R}(u)$ defined on I is said to be of class m, if it has continuous mth order derivative at every point of I.

If we represent $\mathbf{R}$ vectorially as $\mathbf{R} = (x, y, z)$, then the above definition implies that each of the components x, y, z is of class m and x, y, z are functions of u.

Definition 4. If $\dot{\mathbf{R}} = \dfrac{d\mathbf{R}}{du}$ never vanishes on I, then the vector valued function $\mathbf{R} = \mathbf{R}(u)$ is said to be regular. This implies that $\dot{x}$, $\dot{y}$, $\dot{z}$ will never vanish on I simultaneously.

Definition 5. A regular vector valued function $\mathbf{R} = \mathbf{R}(u)$ of class m is called a path of class m.

***Note* 1.** As the parameter u varies, $\mathbf{R}(u)$ gives the position vector of different points on the curve. Thus a path can be considered as the locus of a moving point giving the manner in which the curve is described.

***Note* 2.** Since the definition of a path depends not only on a vector valued function $\mathbf{R}(u)$, but also on the interval I, there are as many paths as there are regular vector valued functions of class m defined on I. Likewise a single path may be defined by different C^m functions defined on different intervals I_1, I_2 etc. For example we can have a path of the same class defined on I_1 and I_2 corresponding to two different vector valued functions. Hence to arrive at a unique single path of the given class corresponding to the parameter u defined on I, it is desirable to partition the paths into mutually disjoint classes of the same type and choose representative path of the same class with a unique parameter. We shall achieve this by an equivalence relation among the paths of the same class as follows.

Definition 6. Two paths $\mathbf{R}_1$ and $\mathbf{R}_2$ of class C^m defined on I_1 and I_2 are said to be equivalent if there exists a strictly increasing function ϕ of class m which maps I_1 onto I_2 such that $\mathbf{R}_1 = \mathbf{R}_2 \circ \phi$

If we take $\mathbf{R}_1 = (x_1, y_1, z_1)$ and $\mathbf{R}_2 = (x_2, y_2, z_2)$, then the above conditions are the same as

$$x_1(u) = x_2(\phi(u)),\ y_1(u) = y_2(\phi(u)),\ z_1(u) = z_2(\phi(u))$$

First let us verify that the notion of equivalence of paths of the same class as defined in the previous paragraph is an equivalence relation.

(i) To prove the relation is reflexive, let $\mathbf{R}_1$ be a path of class m defined on I and let us take $\phi = i$. The identity function is an increasing function on I and $\mathbf{R}_1 = \mathbf{R}_1 \circ i$ so that $\mathbf{R}_1$ is equivalent to itself. Thus the relation of equivalence of paths is reflexive.

(ii) Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the paths of the same class m defined on I_1 and I_2 respectively. Let $\mathbf{R}_1$ be equivalent to $\mathbf{R}_2$. We shall show that $\mathbf{R}_2$ is equivalent to $\mathbf{R}_1$. Since $\mathbf{R}_1$ is equivalent to $\mathbf{R}_2$, there exists a strictly increasing function ϕ from I_1 onto I_2 such that $\mathbf{R}_1 = \mathbf{R}_2 \circ \phi$. Since ϕ is a strictly increasing function on I_1 onto I_2 with $\phi \neq 0$, the inverse function ϕ^{-1} exists as a strictly increasing function on I_2 onto I_1. Hence we have $\mathbf{R}_2 = \mathbf{R}_1 \circ \phi^{-1}$ which shows that $\mathbf{R}_2$ is equivalent to $\mathbf{R}_1$. Hence this relation is symmetric.

(iii) To prove the relation is transitive, let the path $\mathbf{R}_1$ be equivalent to $\mathbf{R}_2$ and $\mathbf{R}_2$ be equivalent to $\mathbf{R}_3$. Then there exists a strictly increasing function ϕ defined on I_1 onto I_2 such that

$$\mathbf{R}_1 = \mathbf{R}_2 \circ \phi. \qquad \ldots(1)$$

In a similar manner, there exists a strictly increasing function ψ on I_2 onto I_3 such that

$$\mathbf{R}_2 = \mathbf{R}_3 \circ \psi. \qquad \ldots(2)$$

Since $\phi: I_1 \to I_2$ and $\psi: I_2 \to I_3$ are strictly increasing functions $\psi \circ \phi$ is a well-defined strictly increasing function on I_1 onto I_3. From (1) and (2) we have $\mathbf{R}_1 = \mathbf{R}_2 \circ \phi = \mathbf{R}_3 \circ \psi \circ \phi$

Since $\psi \circ \phi$ is strictly increasing function on I_1 onto I_3, $\mathbf{R}_1$ is equivalent to $\mathbf{R}_3$, proving that the relation is transitive.

Thus the notion of equivalence of paths of the same class C^m is an equivalence relation. This relation introduces a partition on the paths of the same class splitting the paths of the same class into mutually disjoint classes such that the paths within the same class are equivalent to one another. Using these mutually disjoint classes, we shall define a space curve and the parametric representation as follows.

These different equivalent classes of paths of class m determine the curves of class m. Thus any path $\mathbf{R}$ determines a unique curve and $\mathbf{R}$ is called the parametric representation of the curve. The variable u is called the parameter. Further $\mathbf{R} = (x, y, z)$ where $x = x(u), y = y(u), z = z(u)$ is called the parametric representation of the curve. The function ϕ of two equivalent paths is called the change of parameter. Though ϕ preserves the sense of description of the curve, it gives the changes in the manner of description of the curve. Summarising the above, we define a curve as follows.

Definition 7. Any curve of class m in E_3 is defined to be any set of points in E_3 associated with an equivalence class of regular parametric representations of class m having one parameter.

Since the properties of a curve depend on a particular parameter chosen, every property of a path is not a property of the curve. We are concerned with those properties of a curve which are common to all parametric representations. This means those properties which are invariant under a parametric transformation.

Before proceeding further, we shall illustrate the equivalent representations by the following two examples.

Example 1. The following are the two equivalent representations of a circular helix.

(i) $\mathbf{R}_1(u) = (a \cos u, a \sin u, bu)$, $u \in I_1 = [0, \pi)$

(ii) $\mathbf{R}_2(v) = \left(a \dfrac{1-v^2}{1+v^2}, \dfrac{2av}{1+v^2}, 2b \tan^{-1} v\right)$, $v \in I_2 = [0, \infty)$

To show that $\mathbf{R}_1$ is equivalent to $\mathbf{R}_2$, we find a change of parameter $\phi(u)$ from I_1 to I_2 such that $\mathbf{R}_1(u) = \mathbf{R}_2[\phi(u)]$.

Let us take $v = \phi(u) = \tan \dfrac{1}{2} u$ which is an increasing function from I_1 onto I_2 so that we can take this function as a change of parameter.

Hence $\quad \mathbf{R}_2(v) = \mathbf{R}_2\,[\phi(u)]$

$$= \left[a \frac{1-\tan^2 \frac{1}{2}u}{1+\tan^2 \frac{1}{2}u}, \frac{2a \tan \frac{u}{2}}{1+\tan^2 \frac{u}{2}}, 2b \tan^{-1}\left(\tan \frac{u}{2}\right)\right]$$

$$= (a \cos u, b \sin u, bu) = \mathbf{R}_1(u)$$

Thus there exists a change of parameter $\phi(u)$ such that $\mathbf{R}_1(u) = \mathbf{R}_2[\phi(u)]$ so that $\mathbf{R}_2$ is equivalent to $\mathbf{R}_1$.

Example 2. Let $I_1 = \left(0, \dfrac{\pi}{2}\right)$ and $I_2 = (0, 1)$.

Let $\mathbf{R}_1(u) = (2 \cos^2 u, \sin 2u, 2 \sin u)$ be defined on I_1

If $\phi : I_2 \to I_1$ is defined as $u = \phi(v) = \sin^{-1}(v)$, find the parametr representation $\mathbf{R}_2(v)$ equivalent to $\mathbf{R}_1(u)$.

Now $\phi : I_2 \to I_1$ is $u = \phi(v) = \sin^{-1}(v)$. Then we have $v = \sin u$.

Hence $\quad \mathbf{R}_1(u) = [2(1 - \sin^2 u), 2 \sin u \cos u, 2 \sin u]$

Now $\quad \mathbf{R}_2(v) = \mathbf{R}_1[\phi(v)]$

$\mathbf{R}_2(v) = [2(1 - v^2),\ 2v\sqrt{1-v^2},\ 2v]$ which is the required parametric transformation equivalent to $\mathbf{R}_1(u)$.

1.4 ARC-LENGTH

We define the arc length of a curve and derive a formula for the arc length. Using this formula of arc length, we show that the arc length is invariant under parametric representation so that the arc length of a curve can be used as a parameter in our study of properties of a space curve. Hence the arc length of a space curve plays the important role as a natural parameter.

Definition 1. Let $\mathbf{R} = \mathbf{R}(u)$ be a path with parameter $u \in I$. As u varies over $[a, b] \subset I$, the path is an arc of the original path joining a and b. Let Δ be the subdivision of $[a, b]$ as follows

$$\Delta = \{a = u_0 < u_1 < u_2 < \dots < u_i < \dots < u_n = b\}$$

Corresponding to this subdivision Δ of $[a, b]$, let

$$L(\Delta) = \sum_{i=1}^{n} |\mathbf{R}(u_i) - \mathbf{R}(u_{i-1})|$$

Then $L(\Delta)$ gives the sum of the lengths of the sides of the polygon inscribed within the curve by joining the successive points on the path. Any addition of new points like u_i' in the side $u_{i-1}\, u_i$ increases $L(\Delta)$,
since $|\mathbf{R}(u_i) - \mathbf{R}(u_{i-1})| \le |\mathbf{R}(u_i') - \mathbf{R}(u_{i-1})| + |\mathbf{R}(u_i) - \mathbf{R}(u_i')|$

Since $a \neq b$, as Δ varies over all possible subdivisions of $[a, b]$, we obtain a non-empty subset $\{L(\Delta)\}$ of real numbers. The least upper bound of this set $\{L(\Delta)\}$ of real numbers is defined to be the length of the arc between a and b.

The following theorem asserts the existence of finite upperbound for the set $\{L(\Delta)\}$ and gives a formula for it.

Theorem 1. If $\mathbf{R} = \mathbf{R}(u)$ is the parametric representation of a curve where $u \in [a, b]$, the length of the curve

$$s = S(u) = \int_a^u |\dot{\mathbf{R}}(u)|\, du \qquad \dots(1)$$

Proof. For a subdivision $\Delta = \{a = u_0 < u_1 < u_2 \dots < u_n = b\}$

we have
$$L(\Delta) = \sum_{i=1}^{n} |\mathbf{R}(u_i) - \mathbf{R}(u_{i-1})| \qquad \dots(2)$$

Since $\mathbf{R}$ is at least of class C^1, we have

$$|\mathbf{R}(u_i) - \mathbf{R}(u_{i-1})| = \left|\int_{u_{i-1}}^{u_i} \mathbf{R}(u)\, du\right| \qquad \dots(3)$$

Using (3) in (2), we obtain,

$$L(\Delta) = \sum_{i=1}^{n} \left|\int_{u_{i-1}}^{u_i} \dot{\mathbf{R}}(u)\, du\right|$$

By Schwarz inequality, we get

$$\sum_{i=1}^{n}\left|\int_{u_{i-1}}^{u_i}\dot{\mathbf{R}}(u)\,du\right| \le \sum_{i=1}^{n}\int_{u_{i-1}}^{u_i}|\dot{\mathbf{R}}(u)|\,du = \int_a^b |\dot{\mathbf{R}}(u)|\,du \quad ...(4)$$

so that we have $L(\Delta) \le \int_a^b |\dot{\mathbf{R}}|\,du$

Since the right hand side of (4) is finite and independent of Δ, the set $\{L(\Delta)\}$ for all possible subdivisons Δ of $[a, b]$ is a bounded set of real numbers and it is bounded above. So the least upper bound of $\{L(\Delta)\}$ exists as a finite quantity.

Next we shall show that this upperbound is actually (1) given in the theorem.

If $S = S(u)$ denotes the arc length from a to u, then $S(u) - S(u_0)$ gives the arc length between u_0 and u. Since we have defined the arc length as the least upper bound of $\{L(\Delta)\}$, we have from (4)

$$S(u) - S(u_0) \le \int_{u_0}^{u} |\dot{\mathbf{R}}(u)|\,du \quad ...(5)$$

Since the length of the chord joining $\mathbf{R}(u)$ and $\mathbf{R}(u_0)$ is less than the arcual length, we have

$$|\mathbf{R}(u) - \mathbf{R}(u_0)| \le S(u) - S(u_0) \quad ...(6)$$

From (5) and (6), we have

$$\left|\frac{\mathbf{R}(u) - \mathbf{R}(u_0)}{u - u_0}\right| \le \frac{S(u) - S(u_0)}{u - u_0} \le \frac{1}{u - u_0}\int_{u_0}^{u} |\dot{\mathbf{R}}(u)|\,du \quad ...(7)$$

Taking the limit as $u \to u_0$, we get from (7)

$$|\dot{\mathbf{R}}(u_0)| \le \dot{S}(u_0) \le |\dot{\mathbf{R}}(u_0)|$$

Hence $\dot{S}(u_0)$ exists and has the value $\dot{S}(u_0) = |\dot{\mathbf{R}}(u_0)|$ ...(8)

Since (8) is equally true for any parameter u_0 in I, we conclude from (8)

(i) S is a function of the same class as the curve

(ii) As $S(a) = 0$, $s = S(u) = \int_a^u |\dot{\mathbf{R}}(u)|\,du$ ...(9)

where s denotes the length of the curve from a to u.

Corollary. In terms of cartesian parametric representation,

$$s = \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,du.$$

Proof. In cartesian parametric representation, let $\mathbf{R}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k}$.

Then $|\dot{\mathbf{R}}(u)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Using this expression in (9), we get

$$s = \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,du$$

Further, since $\dot{s} = |\dot{\mathbf{R}}|$, $\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ which gives in terms of differential $ds^2 = dx^2 + dy^2 + dz^2$.

Note. Since $\dot{s} \neq 0$, we can take s as a new parameter. The change of parameter from s to u is given by $S(u)$ in (9). From (9), we can obtain $u = \phi(s)$ so that the curve parametrised with respect to s is $\mathbf{R} = \mathbf{R}[\phi(s)]$.

Theorem 2. The arc length of a curve is invariant under parametric transformation.

Proof. Let $\mathbf{R}_1(u)$ be the parametric representation of the given curve with parameter u. Let us consider the parametric transformation $t = \phi(u)$. Let the parametric representation corresponding to t be $\mathbf{R}_2(t)$. Since $\mathbf{R}_1$ and $\mathbf{R}_2$ are equivalent representations $\mathbf{R}_1(u) = \mathbf{R}_2(t) = \mathbf{R}_2[\phi(u)]$. As u varies from a to b, $t = \phi(u)$ varies from $\phi(a)$ to $\phi(b)$ and $\dot{\mathbf{R}}_1(u) = \dot{\mathbf{R}}_2(t) \cdot \dfrac{dt}{du}$ ...(1)

Now using (1) in the arc-length formula, we get

$$\int_a^b |\dot{\mathbf{R}}_1|\, du = \int_{\phi(a)}^{\phi(b)} |\dot{\mathbf{R}}_2(\mathbf{t})| \left|\frac{dt}{du}\right| \cdot du \qquad ...(2)$$

Since $\mathbf{t} = \phi(u)$ is a strictly increasing function and $\dot{\phi}(u) \neq 0$, we have

$$\left|\frac{dt}{du}\right| = \frac{dt}{du} \qquad ...(3)$$

Using (3) in (2), we obtain $\displaystyle\int_a^b |\dot{\mathbf{R}}_1|\, du = \int_{\phi(a)}^{\phi(b)} |\dot{\mathbf{R}}_2|\, dt$ which proves that the arc length is invariant for a change of parameter from u to t.

Note. When we change the parameter from u to t, the formula for arc length retains its form with t instead of u. This is a very important property of the arc length.

Example 1. Find the arc length of one complete turn of the circular helix

$$\mathbf{r}(u) = (a \cos u, a \sin u, bu), -\infty < u < \infty \qquad ...(1)$$

where $a > 0$ and obtain the equation of the helix with s as parameter.

From (1) $\dot{\mathbf{r}}(u) = (-a \sin u, a \cos u, b)$

So $$s = S(u) = \int_o^u |\dot{\mathbf{r}}(u)|\, du = \int_o^u \sqrt{a^2 + b^2}\, du = cu \qquad ...(2)$$

where $c = \sqrt{a^2 + b^2}$.

If a helix starts from u_0, it makes one complete turn when $u = u_0 + 2\pi$. Hence the arc length corresponding to one complete turn is

$$s = \int_{u_0}^{u_0 + 2\pi} \sqrt{a^2 + b^2}\, du = \sqrt{a^2 + b^2} \cdot 2\pi = 2\pi c.$$

To obtain the equation of the helix with s as parameter,

we have from (2), $s = cu$ so that $u = \dfrac{s}{c}$ ...(3)

Using (3) in (1), we get

$$\mathbf{r}(s) = \left(a\cos\left(\frac{s}{c}\right),\ a\sin\left(\frac{s}{c}\right),\ \frac{bs}{c}\right)$$ which is the required equation.

1.5 TANGENT AND OSCULATING PLANE

If γ is the given curve, then its parametric representation $\mathbf{r} = \mathbf{r}(u)$ is the equation of the curve in the sense that it gives the position vector of different points on γ. We use $\mathbf{R}$ to represent position vectors of points in space not necessarily on γ in order to distinguish it from $\mathbf{r} = \mathbf{r}(u)$ on γ. We also assume γ is of class ≥ 1 which means that $\mathbf{r}(u)$ has continuous derivatives of all orders so that $\mathbf{r} = \mathbf{r}(u)$ has power series expansion at a point u_0 in the neighbourhood u.

$$\mathbf{r}(u_0 + h) = \mathbf{r}(u_0) + \frac{h}{1!}\dot{\mathbf{r}}(u_0) + \frac{h^2}{2!}\ddot{\mathbf{r}}(u_0) + \ldots + \frac{h^n}{n!}\mathbf{r}^{(n)}(u_0) + O(h^n)$$

where $\displaystyle\lim_{h\to 0}\frac{O(h^n)}{h^n} = 0$ where $u - u_0 = h$.

In our study, we usually include first two or three terms in the above expansion.

Definition 1. Unit tangent vector to γ at P. Let P and Q be two neighbouring points on γ with parameters u_0 and u respectively. The parameters of P and Q are very close together in the sense that when $Q \to P$, $u - u_0 \to 0$. The unit vector along PQ tends to the unit vector at P as $Q \to P$. This unit vector denoted by $\mathbf{t}$ is defined to be the unit tangent vector at P. The sense of $\mathbf{t}$ is that of increasing s.

Definition 2. The line through P parallel to $\mathbf{t}$ is called the tangent line to γ at P. If $\mathbf{R}$ is the position vector of any point on this tangent line to γ at P, then the vector $\overrightarrow{PR}$ is called the tangent vector to γ and P.

From the above definition of the tangent vector at P, we have the following properties.

(i) The unit tangent vector $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$ where we have chosen s as the parameter measured from P.

Proof. The unit vector along the chord PQ is

$$\frac{\mathbf{r}(u) - \mathbf{r}(u_0)}{|\mathbf{r}(u) - \mathbf{r}(u_0)|}$$

Since $\mathbf{t}$ is the unit vector along the chord PQ as $Q \to P$,

we get $$\mathbf{t} = \lim_{u\to u_0}\frac{\mathbf{r}(u) - \mathbf{r}(u_0)}{|\mathbf{r}(u) - \mathbf{r}(u_0)|} = \lim_{u\to u_0}\frac{\mathbf{r}(u) - \mathbf{r}(u_0)}{|u - u_0|}\cdot\frac{|u - u_0|}{|\mathbf{r}(u) - \mathbf{r}(u_0)|}$$

Since the curve is of class $m \geq 1$, we get

$$\mathbf{t} = \frac{\dot{\mathbf{r}}(u)}{|\dot{\mathbf{r}}(u)|}$$

Since $\dot{s} = |\dot{\mathbf{r}}(u)|$, we get $\mathbf{t} = \dfrac{\dot{\mathbf{r}}(u)}{\dot{s}(u)} = \dfrac{d\mathbf{r}}{du} \cdot \dfrac{du}{ds} = \dfrac{d\mathbf{r}}{ds}$.

Note. $\dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{du}$ is also a tangent vector, but not necessarily a unit vector.

(ii) Equation of the tangent line at P. Let $\mathbf{R}$ be the position vector of any point Q on the tangent line at P. Then if the length of PQ is c, then the vector $\overrightarrow{PQ} = c\mathbf{t}$. Hence the equation of the tangent line at P is

$$\mathbf{R} = \mathbf{r} + c\mathbf{t}$$

which can also be written as $\mathbf{R} = \mathbf{r} + c\dot{\mathbf{r}}$ where $\dot{\mathbf{r}}$ is parallel to $\mathbf{t}$.

Definition 3. Osculating plane. Let γ be a curve of class $m \geq 2$ and P and Q be two neighbouring points on γ. The limiting position of the plane that contains the tangential line at P and passes through the point Q as $Q \to P$ is defined as the osculating plane at P.

Note. When γ is a straight line the osculating plane is indeterminate at each point. So we avoid this particular case in our discussion.

Definition 4. The point P on the curve for which $\mathbf{r}'' = 0$ is called a point of inflexion and the tangent line at P is called inflexional.

Theorem 1. Let γ be a curve of class $m \geq 2$ with arc length s as parameter. If the point P on γ has parameter zero, the equation of the osculating plane is

$$[\mathbf{R} - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0)] = 0 \text{ where } \mathbf{r}'' \neq 0$$

If $\mathbf{r}''(0) = 0$, let us assume that the curve γ is analytic. Then the equation of the plane at an inflexional point is

$$[R - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}^{(\mathbf{k})}(0)] = 0.$$

Proof. Using the arc length s as parameter, let 0 and s be the parameters of P and Q. Let $\mathbf{R}$ be the position vector of the point on the plane containing the tangent line at P and passing through Q. Then if $\mathbf{r}$ is the position vector of P, then the vectors $\mathbf{R} - \mathbf{r}(0)$, $\mathbf{t} = \mathbf{r}'(0)$, and $\mathbf{r}(s) - \mathbf{r}(0)$ are coplanar vectors. Hence the condition of coplanarity gives the equation as

$$[\mathbf{R} - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}(s) - \mathbf{r}(0)] = 0 \qquad ...(1)$$

Since the curve is of class $m \geq 2$, we have by Taylor's Theorem,

$$\mathbf{r}(s) = \mathbf{r}(0) + s\,\mathbf{r}'(0) + \frac{1}{2}s^2\,\mathbf{r}''(0) + 0(s^2) \text{ as } s \to 0. \qquad ...(2)$$

Using (1) in (2), we get

$$\left[\mathbf{R} - \mathbf{r}(0), \mathbf{r}'(0), s\,\mathbf{r}'(0) + \frac{1}{2}s^2\,\mathbf{r}''(0) + 0(s^2)\right] = 0$$

Neglecting the terms of higher order, the above equation becomes

$$[\mathbf{R}-\mathbf{r}(0), \mathbf{r}'(0), s\,\mathbf{r}'(0)] + \left[\mathbf{R}-\mathbf{r}(0), \mathbf{r}'(0), \frac{s^2}{2!}\mathbf{r}''(0)\right] = 0 \qquad \text{...(3)}$$

Since $\mathbf{r}'(0)\times\mathbf{r}'(0)$ and s is a scalar, the first term of (3) vanishes and so we get

$$[\mathbf{R}-\mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0)] = 0 \qquad \text{...(4)}$$

as the equation of the osculating plane provided the vectors $\mathbf{r}'(0)$ and $\mathbf{r}''(0)$ are linearly independent. So to complete the proof, we have to prove $\mathbf{r}'(0)$ and $\mathbf{r}''(0)$ are linearly independent. Since $\mathbf{t}=\mathbf{r}'$ is a unit vector $\mathbf{r}'^2 = 1$. Differentiating this relation, we have $\mathbf{r}'\cdot\mathbf{r}''=0$ which shows that neither $\mathbf{r}'$ nor $\mathbf{r}''$ can be a constant multiple of the other so that they are linearly independent unless $\mathbf{r}''(0)=0$.

If $\mathbf{r}''(0)=0$, then the point P is an inflexional point so we derive the equation of the osculating plane at an inflexional point with an assumption that the curve γ is analytic.

Differentiating $\mathbf{r}'^2 = 1$, we have $\mathbf{r}'\cdot\mathbf{r}''=0$

Differentiating this relation once again, we have

$$\mathbf{r}''\cdot\mathbf{r}'' + \mathbf{r}'\cdot\mathbf{r}''' = 0$$

Since $\mathbf{r}''=0$, we get $\mathbf{r}'\cdot\mathbf{r}'''=0$ at P.

Since $\mathbf{r}'$ cannot be zero, $\mathbf{r}'$ and $\mathbf{r}'''$ are linearly independent, unless $\mathbf{r}'''=0$. Repeating this process of differentiation, let us assume that $\mathbf{r}^{(\mathbf{k})}$ is first non-vanishing derivative of $\mathbf{r}$ such that $\mathbf{r}'\cdot\mathbf{r}^{(\mathbf{k})}=0$. So if $\mathbf{r}^{(\mathbf{k})}\neq 0$, we have from Taylor Theorem

$$\mathbf{r}(s)-\mathbf{r}(0) = \frac{\mathbf{r}'(0)s}{1!} + \frac{s^k}{k!}\mathbf{r}^{(\mathbf{k})}(0) + 0(s^k) \text{ as } s\to 0. \qquad \text{...(5)}$$

Using (5) in (1), we get

$$\left[\mathbf{R}-\mathbf{r}(0), \mathbf{r}'(0), \frac{\mathbf{r}'(0)}{1!}s + \frac{s^k}{k!}\mathbf{r}^{(\mathbf{k})}(0)\right] = 0$$

As in the previous case the above equation reduces to $[\mathbf{R}-\mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}^{(\mathbf{k})}(0)] = 0$ as the equation of the osculating plane at an inflexional point.

If $\mathbf{r}^{(\mathbf{k})}=0$ for all $k\geq 2$, then since the curve is analytic, we infer that $\mathbf{t}$ is constant and therefore the curve is a straight line.

Corollary. If P is not point of inflexion, any vector lying in the osculating plane is $a\,\mathbf{r}' + b\,\mathbf{r}''$ for some constants a and b.

Proof. Since P is not a point of inflexion $\mathbf{r}''\neq 0$. From (4) of the theorem $\mathbf{r}'$ and $\mathbf{r}''$ lies in the osculating plane and pass through P. Hence any vector in the osculating plane is a linear comlination of $\mathbf{r}'$ and $\mathbf{r}''$ so that we can take it as $a\mathbf{r}' + b\mathbf{r}''$ for some constants a and b. It is of importance to note that $\mathbf{r}''$ lies in the osculating plane.

Theorem 1. If s is the parameter of the curve, then the equation of the osculating plane at any point P with position vector $\mathbf{r} = \mathbf{r}(s)$ is

$$[illegible]$$

Proof. [illegible]

Further [illegible]

Using these values of $\mathbf{r}'$ and $\mathbf{r}''$ in [illegible] of the previous theorem we have

$$[illegible]$$

Since [illegible] and [illegible] is a scalar, simplifying the above equation, we obtain [illegible] as the equation of the osculating plane.

Corollary. If $\mathbf{R} = (X, Y, Z)$ and $\mathbf{r} = (x, y, z)$ then the equation of the osculating plane given by the scalar triple product in the theorem takes the form

$$[illegible]$$

Note 1. A tangent at a point P on a space curve is the line passing through the two consecutive points on the curve. Likewise the osculating plane can be defined as the limiting position of the plane PQR of three consecutive points P, Q, R as Q and R approach P. Using this definition, we shall derive the equation of the osculating plane as follows:

Let $\mathbf{r} = \mathbf{r}(u_0)$, $\mathbf{r}_1 = \mathbf{r}(u_1)$ and $\mathbf{r}_2 = \mathbf{r}(u_2)$ be three consecutive points on the curve. Let $\mathbf{R} \cdot \mathbf{a} = p$ be the equation of the plane passing through the above three points. Then if $f(u) = \mathbf{r} \cdot \mathbf{a} - p$, we have

$$f(u_0) = 0,\ f(u_1) = 0 \text{ and } f(u_2) = 0.$$

Now we have the intervals (u_0, u_1) and (u_1, u_2). Hence by Rolle's Theorem, there exist points $v_1 \in (u_0, u_1)$ and $v_2 \in (u_1, u_2)$ such that $f'(v_1) = 0$ and $f'(v_2) = 0$. Since f' satisfies the conditions of the Rolle's Theorem in (v_1, v_2), there exists $w \in (v_1, v_2)$ such that $f''(w) = 0$. Hence when Q and R approach P, u_1, u_2, v_1, v_2, w approach u_0. Writing u for u_0 in the limiting case, we have

$$f(u) = \mathbf{r} \cdot \mathbf{a} - p = 0,\ f'(u) = \mathbf{r}' \cdot \mathbf{a} = 0,\ f''(u) = \mathbf{r}'' \cdot \mathbf{a} = 0 \quad (1)$$

Using $\mathbf{R} \cdot \mathbf{a} = p$ in the first equation, we get

$$f(u) = (\mathbf{R} - \mathbf{r}) \cdot \mathbf{a} = 0$$

Thus from (1) we find the vector $\mathbf{a}$ is perpendicular to $(\mathbf{R} - \mathbf{r})$, $\mathbf{r}'$ and $\mathbf{r}''$ so that they are coplanar. So we write $\mathbf{R} - \mathbf{r} = \lambda \mathbf{r}' + \mu \mathbf{r}''$ where λ and μ are constants. Eliminating λ, μ or using the condition of coplanarity, the equation of the osculating plane is

$$[\mathbf{R}-\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0$$

Note **2.** In the case of the plane curves, the plane through the three consecutive points on the curve is the plane itself. Hence the osculating plane coincides with the plane of the curve itself.

The following example shows that at a point of inflexion, even a curve of class ∞ need not possess an osculating plane.

Example 1. Let γ be a curve defined by

$$\mathbf{r}(u) = (u, 0, e^{-1/u^2}) \quad \text{when } u > 0$$

$$\mathbf{r}(u) = (u, e^{-1/u^2}, 0) \quad \text{when } u < 0$$

and $$\mathbf{r}(u) = (0, 0, 0) \quad \text{when } u = 0$$

First we shall show that the given curve γ is of class infinity with $u = 0$ as an inflexion point. Taking $f(u) = e^{-1/u^2}$, we prove $f^{(k)}(0) = 0$ for all $k \geq 2$

Now $$f'(0) = \lim_{u\to 0} \frac{f(u)-f(0)}{u} = \lim_{u\to 0} \frac{e^{-1/u^2}}{u} = 0$$

$$f''(0) = \lim_{u\to 0} \frac{f'(u)-f'(0)}{u} = \lim_{u\to 0} \frac{2e^{-1/u^2}}{u^4} = 0$$

$$f'''(0) = \lim_{u\to 0} \frac{1}{u}\left(\frac{4}{u^6}-\frac{6}{u^4}\right)e^{-1/u^2} = \lim_{y\to\infty} \frac{4y^7-6y^5}{e^{y2}} = 0$$

In a similar manner $f^{(k)}(0) = 0$

Now $$\mathbf{r}''(u) = (0, f''(u), 0) \text{ if } u < 0$$

and $$\mathbf{r}''(u) = (0, 0, f''(u)) \text{ if } u > 0$$

Hence when $u \to 0$, $\mathbf{r}''(u) = (0, 0, 0)$ which proves that $u = 0$ is an inflexional point.

Since $f^{(k)}(0) = 0$ for all k, $\mathbf{r}^{(k)}(0) = 0$ for all $k \geq 2$. Hence γ is a curve of class infinity with $u = 0$ as an inflexion point.

Now let us find the equation of the osculating plane when $u > 0$

$$\mathbf{r}(u) = (u, 0, e^{-1/u^2}),\ \mathbf{r}'(u) = \left(1, 0, \frac{2}{u^3}e^{-1/u^2}\right)$$

and $$\mathbf{r}''(u) = \left(0, 0, \left(\frac{4}{u^6}-\frac{6}{u^4}\right)e^{-1/u^2}\right)$$

Hence the equation of the osculating plane is

$$\begin{vmatrix} X-u & Y & Z-e^{-1/u^2} \\ 1 & 0 & \frac{2}{u^3}e^{-1/u^2} \\ 0 & 0 & \left(\frac{4}{u^6}-\frac{6}{u^4}\right)e^{-1/u^2} \end{vmatrix} = 0$$

Expanding the above determinant along the last column,

$$\left(\frac{4}{u^6} - \frac{6}{u^4}\right)e^{-1/u^2}\, Y = 0$$

Since $\left(\frac{4}{u^6} - \frac{6}{u^4}\right)e^{-1/u^2} \neq 0$, the equation of the osculating plane when $u > 0$ is $Y = 0$. In a similar manner the equation of the osculating plane when $u < 0$ is $Z = 0$. So the osculating plane at $u = 0$ is indeterminate. This proves that at a point of inflexion, even a curve of class ∞ need not possess an osculating plane.

Example 2. Find the equation of the osculating plane at a point u of the circular helix

$$\mathbf{r} = (a \cos u,\ a \sin u,\ bu) \qquad \text{...(1)}$$

From the given equation, we have

$$\dot{\mathbf{r}} = (-\,a \sin u,\ a \cos u,\ b) \qquad \text{...(2)}$$

$$\ddot{\mathbf{r}} = (-\,a \cos u,\ -\,a \sin u,\ 0) \qquad \text{...(3)}$$

Using (1), (2), and (3), the equation of the osculating plane is

$$\begin{vmatrix} X - a\cos u & Y - a \sin u & Z - bu \\ -\,a \sin u & a \cos u & b \\ -\,a\cos u & -\,a \sin u & 0 \end{vmatrix} = 0$$

Expanding the above determinant along the last column and simplifying, the equation of the osculating plane is $b(X \sin u - Y \cos u - au) + aZ = 0$.

1.6 PRINCIPAL NORMAL AND BINORMAL

Besides the tangent at P, we shall define the normal and binormal at P of the curve leading to the moving triad of coordinate system at P.

Definition 1. Let P be a point on the curve γ. The plane through P orthogonal to the tangent at P is called the normal plane at P.

Since the osculating plane at P passes through the tangent at P, the normal plane is perpendicular to the osculating plane at any point of the curve.

Definition 2. The line of intersection of the normal plane and the osculating plane is called the principal normal at P. The unit vector along the principal normal is denoted by $\mathbf{n}$. The sense of $\mathbf{n}$ may be chosen arbitrarlly, provided it varies continuously along the curve.

Using the above defnitions, we have

(i) The equation of the normal plane. Let $\mathbf{r} = \mathbf{r}(u)$ be a point on the curve and $\mathbf{R}$ be the position vector of any point on the plane. Then $(\mathbf{R} - \mathbf{r})$ lies in the normal plane. Since $(\mathbf{R} - \mathbf{r})$ is perpendicular to $\mathbf{t}$, we get $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$ as the equation of the normal plane.

(ii) The equation of the principal normal at P. If the position vector of any point P on the curve is $\mathbf{r}$ and if $\mathbf{R}$ is the position vector of any point Q on the normal, then the vector $PQ = \lambda\mathbf{n}$ where λ is a scalar. Then $\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}$ is the equation of the normal at P.

Definition 3. The normal at P orthogonal to the osculating plane is called the binormal at P. The unit vector along the binormal is denoted by $\mathbf{b}$. The sense of the unit vector $\mathbf{b}$ along the binormal is chosen such that $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ form a right handed system of axes.

The behaviour of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ at a point P on the curve is the same as the unit vectors i, j, k, along the coordinate axes. Hence we have

$$\mathbf{b} = \mathbf{t} \times \mathbf{n},\ \mathbf{t} = \mathbf{n} \times \mathbf{b},\ \mathbf{n} = \mathbf{b} \times \mathbf{t}$$

and

$$\mathbf{t} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{t} = 0,\ \mathbf{t} \cdot \mathbf{t} = \mathbf{b} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{n} = 1.$$

Definition 4. The plane containing the tangent and binormal is called the rectifying plane.

From the above definitions, we have

(i) The equation of the rectifying plane. If $\mathbf{r} = \mathbf{r}(u)$ is a point on the curve and $\mathbf{R}$ is the position vector of any point on the rectifying plane, then $(\mathbf{R} - \mathbf{r})$ is in the rectifying plane and orthogonal to $\mathbf{n}$. Hence $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$ is the equation of the rectifying plane.

Note. Since the binormal is orthogonal to the osculating plane, the equation of the osculating plane can be put in the form $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$.

(ii) The equation of the binormal. Let $\mathbf{r} = \mathbf{r}(u)$ be the position vector of any point P on the curve and let $\mathbf{R}$ be the position vector of any point Q on the binormal. Then $PQ = \mu\mathbf{b}$ where μ is a scalar. Then $\mathbf{R} = \mathbf{r} + \mu\mathbf{b}$ is the equation of the binormal at P.

Note. Since $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$ lie in the osculating plane, $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$ is perpendicular to the osculating plane. Since the binormal is perpendicular to the osculating plane, the binormal is parallel to $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$

Since $\dot{\mathbf{r}}$ gives the direction of the tangent and $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$ gives the direction of the binormal $\dot{\mathbf{r}} \times (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})$ gives the direction of the principal normal. Thus we get $\dot{\mathbf{r}} \times (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}} - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\ddot{\mathbf{r}}$ is the direction of the principal normal.

Note. When we take the arc length as parameter, then we have $r'^2 = 1$ and consequently $\mathbf{r}' \cdot \mathbf{r}'' = 0$. Using this in the above formula, the principal normal is parallel to $\mathbf{r}''$.

Summarising the above we conclude that if we choose the arc length as parameter, then $\mathbf{r}'$, $\mathbf{r}''$ and $\mathbf{r}' \times \mathbf{r}''$ give the directions of the tangent, principal normal and binormal at a point on the curve.

Example. Find the directions and equations of the tangent, normal and binormal and also obtain the normal, rectifying and osculating planes at a point on the circular helix

$$\mathbf{r} = \left(a \cos\left(\frac{s}{c}\right), a \sin\left(\frac{s}{c}\right), \frac{bs}{c}\right)$$

Now
$$\mathbf{r}' = \left(\frac{-a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right) \qquad \text{...(1)}$$

$$\mathbf{r}'' = \left(\frac{-a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right) \qquad \text{...(2)}$$

$$\mathbf{r}'\times\mathbf{r}'' = \left(\frac{ab}{c^3}\sin\left(\frac{s}{c}\right), \frac{-ab}{c^3}\cos\left(\frac{s}{c}\right), \frac{a^2}{c^3}\right) \qquad \text{...(3)}$$

(1), (2) and (3) give the directions of the tangent, normal and binormal.

Next let us find the equations of the tangent, normal and binormal.

The equation of the tangent is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$ which gives

$$\frac{X - a\cos\left(\frac{s}{c}\right)}{-a\sin\left(\frac{s}{c}\right)} = \frac{Y - a\sin\left(\frac{s}{c}\right)}{a\cos\left(\frac{s}{c}\right)} = \frac{Z - \frac{bs}{c}}{b}$$

The equation of the normal is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{b}$ which gives

$$\frac{X - a\cos\left(\frac{s}{c}\right)}{-a\cos\left(\frac{s}{c}\right)} = \frac{Y - a\sin\left(\frac{s}{c}\right)}{-a\sin\left(\frac{s}{c}\right)} = \frac{Z - \frac{bs}{c}}{0}$$

The equation of the binormal is $\mathbf{R} = \mathbf{r} + \mu\mathbf{b}$ which gives

$$\frac{X - a\cos\left(\frac{s}{c}\right)}{ab\sin\left(\frac{s}{c}\right)} = \frac{Y - a\sin\left(\frac{s}{c}\right)}{-ab\cos\left(\frac{s}{c}\right)} = \frac{Z - \frac{bs}{c}}{a^2}$$

Let us find the equations of three planes.

The equation of the normal plane is $(\mathbf{R} - \mathbf{r})\cdot\mathbf{t} = 0$ which gives $ac\cos\left(\frac{s}{c}\right)Y - ac\sin\left(\frac{s}{c}\right)X + (Zc - bs)b = 0$. The equation of the rectifying plane is $(\mathbf{R} - \mathbf{r})\cdot\mathbf{b} = 0$, which gives $\cos\left(\frac{s}{c}\right)X + \sin\left(\frac{s}{c}\right)Y - a = 0$.

The equation of the osculating plane is $(\mathbf{R} - \mathbf{r})\cdot\mathbf{b} = 0$. Since $\mathbf{b}$ has the direction $\mathbf{r}'\times\mathbf{r}''$, we get the above equation as $(\mathbf{R} - \mathbf{r})\cdot(\mathbf{r}'\times\mathbf{r}'') = 0$ ...(4)

Writing (4) in the determinant form, we obtain

$$\begin{vmatrix} X - a\cos\left(\frac{s}{c}\right) & Y - a\sin\left(\frac{s}{c}\right) & Z - \frac{bs}{c} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\frac{a}{c^2}\cos\left(\frac{s}{c}\right) & \frac{-a}{c^2}\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} = 0$$

Expanding the determinant and simplifying, we have

$$b\sin\left(\frac{s}{c}\right)X - b\cos\left(\frac{s}{c}\right)Y + \left(Z - \frac{bs}{c}\right)a = 0$$

which is the equation of the osculating plane.

1.7 CURVATURE AND TORSION

At each point of the curve, we have defined an orthogonal triad **t**, **n**, **b** forming a right handed system and also we have noted that at each point this moving triad determines three fundamental planes which are mutually perpendicular.

It is important to note that **t**, **n**, **b** vary from point to point on the curve as the point moves on the curve as shown in the following figure.

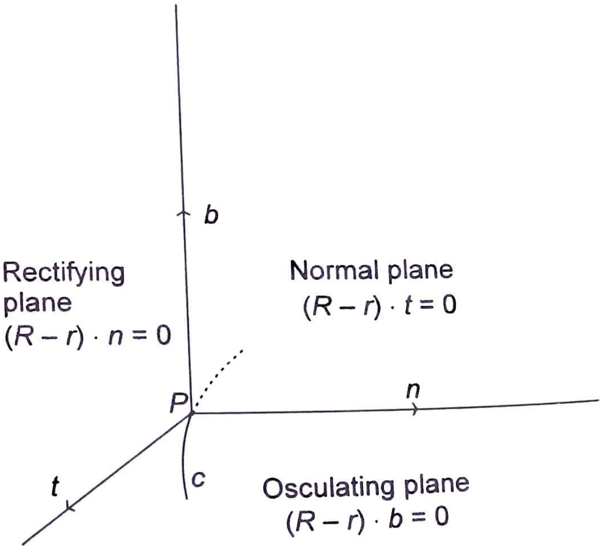

Fig. 1

Hence we can study their variations from point to point with respect to the arcual length as parameter. This leads to the notion of curvature and torsion of the space curve as defined below.

Definitoin 1. The arc rate at whch the tangent changes direction as the point P moves along the curve is called the curvature vector of the curve and it is denoted by K

Thus by definition $\mathbf{K} = \dfrac{d\mathbf{t}}{ds}$ and K is called the curvature vector and its magnitude $|\mathbf{K}|$ is called the curvature at P denoted by κ. Then $\rho = \dfrac{1}{\kappa}$ is called the radius of curvature where we take the absolute value of ρ.

As we have already noted, the osculating plane of a plane curve is the same at all points but the osculating plane changes from point to point on a space curve. So we shall now define a quantity measuring the arc-rate of change of the osculating plane.

Definition 2. The torsion at a point P of a curve is defined as the arc-rate at which the osculating plane turns about the tangent at P as P moves along the curve. It is denoted by τ. $|1/\tau|$ denoted σ is called the radius of torsion.

Having defined the torsion at a point on the space curve, the next question is to devise a method of measuring it. The natural method of measuring the turning of a plane is to measure the turning of its normal. Since the binormal is orthogonal to the osculating plane, we shall use the binormal to measure torsion. Thus the arc-rate of rotation of the osculating plane is expressed by $\mathbf{b}' = \dfrac{d\mathbf{b}}{ds}$ whose magnitude is the torsion τ. So $\tau = \left|\dfrac{d\mathbf{b}}{ds}\right|$.

As a synthesis of the above definitions of curvature and torsion giving the measure of arc-rate of change of tangent and osculating plane, we shall derive Serret-Frenet formulae which are fundamental in the study of space curves.

Theorem 1. (Serret-Frenet Formulae). If $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is the moving orthogonal triad of unit vectors at a point P on a space curve γ, then

$$\text{(i)}\ \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \quad \text{(ii)}\ \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t} \quad \text{(iii)}\ \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}.$$

Proof. We first prove (i) and (iii) and then derive (ii) from them.

To prove (i), differentiating $\mathbf{t}\cdot\mathbf{t} = 1$ with respcet to s at a point P of the curve, we have $\mathbf{t}\cdot\mathbf{t}' = 0$ so that $\mathbf{t}'$ is perpendicular to $\mathbf{t}$.

Since $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$, $\mathbf{t}' = \mathbf{r}''$. As $\mathbf{r}''$ lies in the osculating plane, $\mathbf{t}'$ also lies in the osculating plane. Therefore $\mathbf{t}'$ is a vector perpendicular to $\mathbf{t}$ and lies in the osculating plane. Hence $\mathbf{t}'$ is parallel to the principal normal. By definition $|\mathbf{t}'| = \kappa$, being the curvature at P on the curve. Since we know the magnitude κ and the direction $\mathbf{n}$ of $\mathbf{t}'$, we can write $\mathbf{t}' = \pm\,\kappa\mathbf{n}$. By convention we take $\mathbf{t}' = \kappa\mathbf{n}$.

(ii) As in the above case, we find the vector $\mathbf{b}'$. Differentiating $\mathbf{b}\cdot\mathbf{b} = 1$, we have $\mathbf{b}\cdot\mathbf{b}' = 0$ so $\mathbf{b}'$ is perpendicular to the binormal at P and hence $\mathbf{b}'$ lies in the osculating plane. Since $\mathbf{b}\cdot\mathbf{t} = 0$, differentiating this and using (i) we get $\mathbf{b}'\cdot\mathbf{t} + \mathbf{b}\cdot(\kappa\mathbf{n}) = 0$ As $\mathbf{b}\cdot\mathbf{n} = 0$, we get $\mathbf{b}'\cdot\mathbf{t} = 0$ showing that $\mathbf{b}'$ is perpendicular to $\mathbf{t}$. Therefore $\mathbf{b}'$ is a vector perpendicular to $\mathbf{t}$ and lies in the osculating plane. Hence b′ is parallel to the principal normal at P. By definition b′ = [illegible] P. Since we know the magnitude τ and the direction n of b′, we can uniquely [illegible] where the negative sign is introduced because as a convention torsion is [illegible] as positive when the rotation of the osculating plane is in [illegible] direction of a right handed screw moving in the direction of t.

To prove (iii), let us consider n = b × t.

Differentiating both sides of the above vector with respect to s,

we obtain $\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds}$

Using (i) and (ii) in the above equations, $\frac{d\mathbf{n}}{ds} = (-\tau\mathbf{n}) \times \mathbf{t} + \mathbf{b} \times (\kappa\mathbf{n})$

Since n × t = − b and b × n = − t, we have

$$\frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t}$$

Using curvature and torsion, we characterise a straight line and a plane curve in the next two theorems.

Theorem 2. A necessary and sufficient condition for a curve to be a straight line is that κ = 0 at all points of the curve.

Proof. The condition is necessary. Let us take the curve to be a straight line and let its vector equation be $\mathbf{r} = \mathbf{a}s + \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. Differentiating this equation, we get $\mathbf{r}' = \mathbf{t} = \mathbf{a}$.

As a is a constant vector, we have $\mathbf{r}'' = \mathbf{t}' = 0$. Since the curvature vector t′ vanishes at all points of the curve, its magnitude κ = 0 at all points of the curve.

To prove the sufficiency of the condition, let us assume that κ = 0 at all points of the curve. This implies that the curvature vector $\mathbf{t}' = \mathbf{r}'' = 0$. Integrating the above equation twice, we obtain $\mathbf{r} = \mathbf{a}s + \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. This proves that the curve is a straight line.

Theorem 3. A necessary and sufficient condition that a given curve is a plane curve is that τ = 0 at all points of the curve.

Proof. The condition is necessary. Let us take the curve to be a plane curve and show that τ = 0. Since the curve lies in a plane, the osculating plane at every point of the curve is the plane containing the curve itself so that the binormal b is constant. Since b is constant $\frac{d\mathbf{b}}{ds} = 0$ which implies $\left|\frac{d\mathbf{b}}{ds}\right| = 0$. Hence $\tau = \left|\frac{d\mathbf{b}}{ds}\right|$ = 0 at all points of the curve.

Conversely let τ = 0 at all points of the curve. On this assumption, we prove that the curve is a plane curve. Since τ = 0, $\frac{d\mathbf{b}}{ds} = 0$ at all points of the curve so that b is a constant vector. Now for any vector r,

$$\frac{d}{ds}(\mathbf{r} \cdot \mathbf{b}) = \frac{d\mathbf{r}}{ds} \cdot \mathbf{b} + \mathbf{r} \cdot \frac{d\mathbf{b}}{ds} = \mathbf{t} \cdot \mathbf{b} + \mathbf{r} \cdot 0$$

Since $\mathbf{t}\cdot\mathbf{b}=0$ and $\mathbf{b}'=0$, we have $\dfrac{d}{ds}(\mathbf{r}\cdot\mathbf{b})=0$ for any point $\mathbf{r}$ on the curve. Hence $\mathbf{r}\cdot\mathbf{b}=\text{constant}=c$ (say). If $\mathbf{r}=(x(s),y(s),z(s))$ and $\mathbf{b}=(b_1,b_2,b_3)$, then $\mathbf{r}\cdot\mathbf{b}=c$ gives $xb_1+yb_2+zb_3=c$ which shows that $\mathbf{r}(s)=(x(s),y(s),z(s))$ lies on the plane $b_1X+b_2Y+b_3Z=c$. This proves that the curve is a plane curve and the condition is sufficient.

Definition 3. If τ is non-zero, then the curve is called a twisted curve. In the following, we shall give the formulae for curvature and torsion first in terms of the arc length and then in terms of a general parameter u. We shall use dashes to denote differentiation with respect to s and dots to denote differentiation with respect to u.

Theorem 4. If $\mathbf{r}=\mathbf{r}(s)$ is the position vector of a point P with arc-length as parameter on a curve c, then

(i) $\kappa^2=\mathbf{r}''\cdot\mathbf{r}''$

(ii) $\tau=\dfrac{[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']}{\mathbf{r}''\cdot\mathbf{r}''}$ or $\kappa^2\tau=[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']$

Proof. (i) We know that $\mathbf{r}'=\mathbf{t}$ and $\mathbf{r}''=\kappa\mathbf{n}$ ...(1)

Hence $\mathbf{r}''\cdot\mathbf{r}''=(\kappa\mathbf{n})\cdot(\kappa\mathbf{n})=\kappa^2$, since $\mathbf{n}\cdot\mathbf{n}=1$

(ii) Now $\mathbf{r}'\times\mathbf{r}''=\mathbf{t}\times\kappa\mathbf{n}=\kappa\mathbf{b}$ ...(2)

Differentiating both sides of (2) with respect to s,

$$\mathbf{r}'\times\mathbf{r}'''+\mathbf{r}''\times\mathbf{r}''=\kappa'\mathbf{b}+\kappa\mathbf{b}' \qquad ...(3)$$

Since $\mathbf{r}''\times\mathbf{r}''=0$ and $\mathbf{b}'=-\tau\mathbf{n}$, (3) becomes

$$\mathbf{r}'\times\mathbf{r}'''=(\kappa'\mathbf{b}-\kappa\tau\mathbf{n}) \qquad ...(4)$$

Taking dot product with $\mathbf{r}''$ on both sides of (4), we have

$$(\mathbf{r}'\times\mathbf{r}''')\cdot\mathbf{r}''=-\kappa^2\tau \text{ as } \mathbf{r}''=\kappa\mathbf{n} \text{ and } \mathbf{n}\cdot\mathbf{n}=1 \qquad ...(5)$$

But $(\mathbf{r}'\times\mathbf{r}''')\cdot\mathbf{r}''=-\mathbf{r}'\cdot(\mathbf{r}''\times\mathbf{r}''')=-[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']$ ...(6)

Using (6) in (5), we have $\kappa^2\tau=[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']$ ...(7)

Further substituting for κ^2 from (i), we obtain

$$\tau=\frac{[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']}{\mathbf{r}''\cdot\mathbf{r}''}$$

Theorem 5. If $\mathbf{r}=\mathbf{r}(u)$ is the equation of the curve with parameter u, then

(i) $\kappa=\dfrac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$ and (ii) $\tau=\dfrac{[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}$

Proof. Now $\dot{\mathbf{r}}=\dfrac{d\mathbf{r}}{du}=\dfrac{d\mathbf{r}}{ds}\cdot\dfrac{ds}{du}=\mathbf{r}'\cdot\dot{s}=\mathbf{t}\dot{s}$ ...(1)

Since $\mathbf{t}$ is a unit vector, (1) gives $|\dot{\mathbf{r}}|=\dot{s}$ ...(2)

Differentiating (1) again with respect to u,

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{du^2} = \frac{d}{du}(\mathbf{t}\dot{s}) = \frac{d}{ds}(\mathbf{t}\dot{s})\frac{ds}{du} = \mathbf{t}'\dot{s}^2 + \mathbf{t}\ddot{s}$$

Since $\mathbf{t}' = \kappa\mathbf{n}$, $\ddot{\mathbf{r}} = \kappa\mathbf{n}\dot{s}^2 + \mathbf{t}\ddot{s}$ ...(3)

Using (1) and (3), we get $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \mathbf{t}\dot{s} \times (\kappa\mathbf{n}\dot{s}^2 + \mathbf{t}\ddot{s})$

Since $\mathbf{t} \times \mathbf{n} = \mathbf{b}$ and $\mathbf{t} \times \mathbf{t} = 0$, we get

$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \dot{s}^3\kappa\mathbf{b}$ whch gives ...(4)

the formula for κ as $\kappa = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{\dot{s}^3} = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$ by (2)

To obtain torsion, let us differentiate (4) with respect to u

Then $\dot{\mathbf{r}} \times \dddot{\mathbf{r}} + \ddot{\mathbf{r}} \times \ddot{\mathbf{r}} = (\dot{s}^3\kappa)\,\mathbf{b}'\dot{s} + \mathbf{b}\dfrac{d}{du}(\dot{s}^3\kappa)$

Since $\ddot{\mathbf{r}} \times \ddot{\mathbf{r}} = 0$ and $\mathbf{b}' = -\tau\mathbf{n}$, we obtain

$$\dot{\mathbf{r}} \times \dddot{\mathbf{r}} = -\dot{s}^4\kappa\tau\mathbf{n} + \mathbf{b}\frac{d}{du}(\dot{s}^3\kappa) \qquad ...(5)$$

Taking dot product of (3) and (5),

$$[\ddot{\mathbf{r}}, \dot{\mathbf{r}}, \dddot{\mathbf{r}}] = (\kappa\mathbf{n}\dot{s}^2 + \mathbf{t}\ddot{s}) \cdot \left(-\dot{s}^4\kappa\tau\mathbf{n} + \mathbf{b}\frac{d}{du}(\dot{s}^3\kappa)\right)$$

Since $\mathbf{t} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{b} = \mathbf{t} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$, we obtain

$$[\ddot{\mathbf{r}}, \dot{\mathbf{r}}, \dddot{\mathbf{r}}] = -\dot{s}^6\kappa^2\tau$$

From the property of the scalar triple product,

$[\ddot{\mathbf{r}}, \dot{\mathbf{r}}, \dddot{\mathbf{r}}] = -[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]$ so that $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \dot{s}^6\kappa^2\tau$

Therefore $\tau = \dfrac{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]}{\dot{s}^6\kappa^2} = \dfrac{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}$ using κ.

Note. Using Theorem 4, we obtain the formula for $\kappa^2\tau$ in terms of the dot derivatives as follows.

(i) $\mathbf{r}' = \dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{du} \cdot \dfrac{du}{ds} = \dot{\mathbf{r}}u'$

(ii) $\mathbf{r}'' = \dfrac{d}{ds}(\dot{\mathbf{r}}u') = \dot{\mathbf{r}}u'' + \ddot{\mathbf{r}}u'^2$

(iii) $\mathbf{r}''' = \dfrac{d}{ds}(\dot{\mathbf{r}}u'' + \ddot{\mathbf{r}}u'^2) = \dot{\mathbf{r}}u''' + 3\,\ddot{\mathbf{r}}u'u'' + \dddot{\mathbf{r}}u'^3$

Now from the above equations $\mathbf{r}' \times \mathbf{r}'' = \dot{\mathbf{r}} \times \ddot{\mathbf{r}}u'^3$

Hence $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}u'^3) \cdot (\dot{\mathbf{r}}u''' + 3\,\ddot{\mathbf{r}}u'u'' + \dddot{\mathbf{r}}u'^3)$

$= (u')^6\,[\dot{\mathbf{r}} \times \ddot{\mathbf{r}}] \cdot \dddot{\mathbf{r}} = (u')^6\,[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]$

From Theorem 4, $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = \kappa^2\tau$ so that $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \kappa^2\tau(u')^{-6}$

Further we obtain $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = (u')^6\,[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]$

Corollary. A necessary and sufficient condition that a curve is a plane curve is $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = 0$.

Proof. To prove the necessity of the condition, let us assume that the curve is a plane curve. Then $\tau = 0$ by Theorem 3. Since $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \kappa^2\tau$, $\tau = 0$ implies $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = 0$.

To prove the converse, let $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = 0$. Then $\kappa^2\tau = 0$. Hence either $\kappa = 0$ or $\tau = 0$. We shall prove that $\tau = 0$ at all points of the curve. Let $\tau \neq 0$ at some point of the curve. Then $\tau \neq 0$ in some neighbourhood of that point. Since $\kappa = 0$ in this neighbourhood, the arc of the curve in this neighbourhood is a straight line. This implies that $\tau = 0$ on this line in the neighbourhood of this point contradicting the hypothesis that $\tau \neq 0$. This contradiction proves that $\tau = 0$ at all points of the curve so that the curve is a plane curve. Thus the condition is sufficient also.

Note. In terms of the dash derivatives, we have $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = u'^6\,[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]$. Since $u' = \dfrac{du}{ds} \neq 0$, $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = 0$ if and only if $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = 0$ which is the necessary and sufficient condition for a space curve to be a plane curve.

Example 1. Find the curvature and torsion of the circular helix

$$\mathbf{r} = (a\cos u, a\sin u, bu).$$

From the equation of the helix, we have

$$\begin{aligned}
\dot{\mathbf{r}} &= (-a\sin u, a\cos u, b)\\
\ddot{\mathbf{r}} &= (-a\cos u, -a\sin u, 0)\\
\dddot{\mathbf{r}} &= (a\sin u, -a\cos u, 0)\\
\dot{\mathbf{r}} \times \ddot{\mathbf{r}} &= (ab\sin u, -ab\cos u, a^2)
\end{aligned}$$

Hence $$|\dot{\mathbf{r}}|^2 = a^2 + b^2,\ |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = a^2(a^2 + b^2)$$

Now using the above quantities,

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{a}{(a^2 + b^2)}$$

Now $$(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}} = a^2 b \text{ so that } [\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = a^2 b$$

Hence $$\tau = \frac{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{b}{a^2 + b^2}$$

Note. Sometimes, it is easier to deal with dash formula for curvature and torsion than the dot formula. To illustrate this, we find curvature and torsion of the circular helix by dash formula.

Using the arc-length as parameter, position vector of any point on the circular helix is

$$\mathbf{r} = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right), c^2 = a^2 + b^2$$

$$\mathbf{r}' = \left(\frac{-a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$$

$$\mathbf{r}'' = \left(\frac{-a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$$

$$\mathbf{r}''' = \left(\frac{a}{c^3}\sin\left(\frac{s}{c}\right), \frac{-a}{c^3}\cos\left(\frac{s}{c}\right), 0\right)$$

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ab}{c^3}\sin\left(\frac{s}{c}\right), \frac{-ab}{c^3}\cos\left(\frac{s}{c}\right), \frac{a^2}{c^3}\right)$$

Hence the curvature $\kappa^2 = \mathbf{r}'' \cdot \mathbf{r}'' = \dfrac{a^2}{c^4}$ so that $\kappa = \dfrac{a}{a^2 + b^2}$

Further $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = \dfrac{ba^2}{c^6}$ so that $\tau = \dfrac{[\mathbf{r}', \mathbf{r}''\, \mathbf{r}''']}{\mathbf{r}'' \cdot \mathbf{r}''} = \dfrac{b}{a^2 + b^2}$

Thus we see that κ and τ are constants and $\dfrac{\kappa}{\tau} = \dfrac{a}{b}$. So the ratio of curvatrure to torsion is a constant. This is the characteristic property of a helix which we shall establish later.

Example 2. Calculate the torsion and curvature of the cubic curve

$$\mathbf{r} = (u, u^2, u^3) \qquad \text{...(1)}$$

We shall find κ and τ without using their formulae.

Now from (1) $\dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{ds} \cdot \dfrac{ds}{du} = \mathbf{t}\dot{s} = (1, 2u, 3u^2)$ ...(2)

Hence $\dot{\mathbf{r}}^2 = \mathbf{t}\dot{s} \cdot \mathbf{t}\dot{s} = (1 + 4u^2 + 9u^4)$ which gives

$$\dot{s}^2 = (1 + 4u^2 + 9u^4) \qquad \text{...(3}$$

Differentiating (2), we have

$$\ddot{s}\mathbf{t} + \dot{s}^2\frac{d\mathbf{t}}{ds} = \ddot{s}\mathbf{t} + \dot{s}^2\kappa\mathbf{n} = (0, 2, 6u) \qquad \text{...(4}$$

Now taking the cross product of (2) and (4),

$$\dot{s}^3\kappa\mathbf{b} = 2(3u^2, -3u, 1) \qquad \text{...(}$$

Differentiating (5) with respect to u again

$$\mathbf{b}\frac{d}{du}(\dot{s}^3\kappa) - \dot{s}^4\kappa\tau\mathbf{n} = 2(6u, -3, 0) \qquad \text{...(6)}$$

Taking dot product of (4) and (6), we get

$$(\ddot{s}\mathbf{t} + \dot{s}^2\kappa\mathbf{n})\cdot\left(\mathbf{b}\frac{d}{du}(\dot{s}^3 k) - \dot{s}^4\kappa\tau\mathbf{n}\right) = -12$$

Since $\mathbf{b}\cdot\mathbf{t} = \mathbf{b}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{t} = 0$ and $\mathbf{n}\cdot\mathbf{n} = 1$, we get

$$-\dot{s}^6\kappa^2\tau = -12 \text{ so that } \kappa^2\tau = \frac{12}{\dot{s}^6} \qquad \text{...(7)}$$

Taking dot product of (5) with itself on both sides and using (3)

$$\kappa^2 = \frac{4(9u^4 + 9u^2 + 1)}{(1 + 4u^2 + 9u^4)^3} \qquad \text{...(8)}$$

Using (8) in (7), we get $\tau = \dfrac{3}{(9u^4 + 9u^2 + 1)}$

Example 3. Find the curvature and torsion of

$$\mathbf{r} = (a\cos\theta, a\sin\theta, a\theta\cot\alpha) \qquad \text{...(1)}$$

Instead of using the formulae for κ and τ, we shall find κ and τ from the derivatives of (**t**, **n**, **b**) with the help of Serret-Frenet formulae.

Differentiating (1) with respect to s, we get

$$\frac{d\mathbf{r}}{ds}\cdot\frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, a\cot\alpha)$$

That is $$\mathbf{t}\frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, a\cot\alpha) \qquad \text{...(2)}$$

Taking dot product on both sides of (2) with itself.

$$\left(\frac{ds}{d\theta}\right)^2 = a^2 + a^2\cot^2\alpha \text{ so that } \frac{ds}{d\theta} = a\operatorname{cosec}\alpha$$

From (2) we have $\mathbf{t} = \sin\alpha(-\sin\theta, \cos\theta, \cot\alpha)$ ...(3)

Differentiating (3) with respect to s,

$$\frac{d\mathbf{t}}{ds}\frac{ds}{d\theta} = \sin\alpha(-\cos\theta, -\sin\theta, 0)$$

Using Serret-Frenet formula, we get

$$\kappa\mathbf{n} = \frac{\sin^2\alpha}{a}(-\cos\theta, -\sin\theta, 0) \qquad \text{...(4)}$$

Taking dot product on both sides of (4) with itself,

$$\kappa^2 = \frac{\sin^4 \alpha}{a^2} \quad \text{or } \kappa = \frac{\sin^2 \alpha}{a}$$

Using this value of κ in (4), $\mathbf{n} = (-\cos\theta, -\sin\theta, 0)$

Since we know **t** and **n**, we have

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = (\cos\alpha \sin\theta, -\cos\alpha\cos\theta, \sin\alpha)$$

Hence $$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} = (\cos\alpha\cos\theta, \sin\theta\cos\alpha, 0)\frac{d\theta}{ds} \qquad \text{...(4)}$$

Taking dot product on both sides of (4) with itself,

$$\tau^2 = (\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha)\frac{\sin^2\alpha}{a^2} \text{ giving } \tau \text{ as } \tau = \frac{\sin\alpha\cos\alpha}{a}.$$

1.8 BEHAVIOUR OF A CURVE NEAR ONE OF ITS POINTS

Using Serret-Frenet formula, we shall study the behaviour of a curve in the neighbourhood of a point on the curve. In the following theorem, we obtain the coordinates of a point near the given point on the curve with reference to the coordinate axes along **t**, **n**, **b** and then deduce a number of properties from them.

Theorem 1. Let the curve be of class $m \geq 4$. At a point P on the curve, let the coordinate axes, ox, oy, oz be taken along **t**, **n**, **b**. If X, Y, Z are the coordinates of the neighbouring point Q on the curve, then

$$X = s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa'}{8}s^4 + o(s^4)$$

$$Y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa\tau^2 - \kappa^3}{24}s^4 + o(s^4)$$

$$Z = \frac{\kappa\tau}{6}s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24}s^4 + o(s^4) \text{ as } s \to 0$$

Proof. Since the curve is of class ≥ 4, we have by Taylor's Theorem,

$$\mathbf{r}(s) = \mathbf{r}(0) + \frac{\mathbf{r}'(0)}{1!}s + \frac{\mathbf{r}''(0)}{2!}s^2 + \frac{\mathbf{r}'''(0)}{3!}s^3 + \frac{\mathbf{r}^{(iv)}(0)}{4!}s^4 + o(s^4) \text{ as } s \to 0 \qquad \text{...(1)}$$

where s is the small arc PQ and $\mathbf{r}(0) = 0$

To study the equation (1), let us find $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{r}'''$ and $\mathbf{r}^{(iv)}$ at the origin 0.

(i) $\mathbf{r}'(0) = \mathbf{t}$

(ii) $\mathbf{r}''(0) = \mathbf{t}' = \kappa\mathbf{n}$

(iii) $\mathbf{r}'''(0) = \kappa'\mathbf{n} + \kappa\mathbf{n}' = \kappa'\mathbf{n} + \kappa(\tau\mathbf{b} - \kappa\mathbf{t})$

$= -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}$

(iv) $\mathbf{r}^{(\text{iv})}(0) = -2\kappa\kappa'\mathbf{t} - \kappa^2(\kappa\mathbf{n}) + \kappa''\mathbf{n} + \kappa'(\tau\mathbf{b} - \kappa\mathbf{t}) + \kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} + \kappa\tau(-\tau\mathbf{n})$

$= -3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa\tau^2 - \kappa^3)\mathbf{n} + (2\kappa'\tau + \kappa\tau')\mathbf{b}.$

Using (i), (ii), (iii) and (iv), the Taylor expansion (i) near $s = 0$ becomes

$$\mathbf{r}(s) = \mathbf{r}(0) + \frac{\mathbf{t}s}{1!} + \frac{\kappa\mathbf{n}s^2}{2!} + \{-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}\}\frac{s^3}{3!}$$

$$+ \{-3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa\tau^2 - \kappa^3)\mathbf{n} + (2\kappa'\tau + \kappa\tau')\mathbf{b}\}\frac{s^4}{4!} + o(s^4) \quad ...(2)$$

Noting $\mathbf{r}(0) = 0$ at P and gathering the coefficients of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ in (2) we get

$$\mathbf{r}(s) = \left[s - \frac{\kappa^2}{3!}s^3 - 3\kappa\kappa'\frac{s^4}{4!} + o(s^4)\right]\mathbf{t}$$

$$+ \left[\frac{\kappa}{2!}s^2 + \frac{\kappa'}{3!}s^3 + \frac{(\kappa'' - \kappa\tau^2 - \kappa^3)}{4!}s^4 + o(s^4)\right]\mathbf{n}$$

$$+ \left[\frac{\kappa\tau}{3!}s^3 + \frac{2\kappa'\tau + k\tau'}{24}s^4 + o(s^4)\right]\mathbf{b}$$

If X, Y, Z are the coordinates of the neighbouring point Q with position vector $\mathbf{r}(s)$ with reference to the coordinate system $0(x, y, z)$ in the direction of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, then

$$X = s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa'}{8}s^4 + o(s^4)$$

$$Y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa\tau^2 - \kappa^3}{24}s^4 + o(s^4)$$

$$Z = \frac{\kappa\tau}{6}s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24}s^4 + o(s^4)$$

express the coordinates X, Y, Z in terms of the arcual length $PQ = s$. (X, Y, Z) is called Serret-Frenet approximation of the curve.

Using the above equation, we have the following deductions.

(i) $\dfrac{2Y}{X^2} \sim \kappa$ as $s \to 0$ and $\dfrac{3Z}{XY} \sim \tau$ as $s \to 0$.

Proof. Neglecting powers of s^3, we have $X \sim s$ and $Y \sim \dfrac{\kappa}{2}s^2$.

Eliminating s, $\dfrac{2Y}{X^2} \sim \kappa$ as $s \to 0$.

Neglecting the powers of s^4, we have

$$Z \sim \frac{\kappa\tau}{6}s^3 \text{ and } XY \sim \frac{\kappa}{2}s^3$$

Eliminating s^3, we obtain $Z \sim \frac{\tau}{3}XY$ so that $\tau \sim \frac{3Z}{XY}$

Note. The above formula for κ resembles Newton's formula for curvature of plane curves.

(ii) The chord $PQ = (X^2+Y^2+Z^2)^{1/2} \sim s\left(1-\frac{\kappa^2s^2}{24}\right)$

Proof. Neglecting the powers of s^4, we get the coordinates as

$$X \sim s-\frac{\kappa^2s^3}{6},\ Y \sim \frac{\kappa}{2}s^2+\frac{\kappa's^3}{6},\ Z = \frac{\kappa\tau}{6}s^3 \text{ as } s\to 0$$

Hence $X^2+Y^2+Z^2 = \left(s-\frac{\kappa^2s^3}{6}\right)^2+\left(\frac{\kappa}{2}s^2+\frac{\kappa's^3}{6}\right)^2+\frac{\kappa^2\tau^2}{6^2}s^6$

Neglecting the terms of degree ≥ 5,

$$(X^2+Y^2+Z^2) = s^2-\frac{1}{12}\kappa^2s^4 = s^2\left(1-\frac{1}{12}\kappa^2s^2\right)$$

Thus $\quad (X^2+Y^2+Z^2)^{1/2} = s\left(1-\frac{1}{12}\kappa^2s^2\right)^{1/2}$

Using the Binomial expansion on the right hand side,

$(X^2+Y^2+Z^2)^{1/2} \sim s\left(1-\frac{1}{24}\kappa^2s^2\right)$ which shows that when $\kappa \neq 0$, the arc length PQ differs from the chord PQ by a term of the third order in s.

(iii) We obtain the approximate equations of the projections of the curve on three planes. In each case we obtain the lowest power of s and eliminate s.

(a) The projection of the curve on the osculating plane is $Y = \frac{\kappa}{2}X^2, Z=0$

From the equations, we obtain as in (i) $X \sim s$ and $Y \sim \frac{\kappa}{2}s^2$. Eliminating

between the two equations $Y = \frac{\kappa}{2}X^2$.

(b) The projection of the curve on the rectifying plane is

$$Z = \frac{\kappa\tau}{6}X^3,\ Y=0$$

From the equations, we obtain as in (i) $X \sim s$ and $Z \sim \dfrac{\kappa\tau}{6}s^3$.

Eliminating s between them, $Z = \dfrac{\kappa\tau}{6}X^3$

(c) The projection on the normal plane is $Z^2 = \dfrac{2}{9}\dfrac{\tau^2}{\kappa}Y^3, X = 0$

From the equations, we obtain as in (i), $Z \sim \dfrac{\kappa\tau s^3}{6}$ and $Y = \dfrac{\kappa}{2}s^2$

Eliminating s between the above two equations,

$$Z^2 \sim \frac{\kappa^2\tau^2}{6\cdot 6}\cdot s^6 = \frac{\kappa^2\tau^2}{6\cdot 6}\left(\frac{2Y}{\kappa}\right)^3 = \frac{2}{9}\frac{\tau^2}{\kappa}Y^3$$

Example 1. Find the Serret-Frenet approximation of the curve

$$(\cos u, \sin u, u) \text{ at } u = \frac{\pi}{2}.$$

As in Examples 1 of 1.4 and 1.7, $s = \sqrt{2}u$, $\kappa = \tau = \dfrac{1}{2}$ and $s = \dfrac{\pi}{\sqrt{2}}$ at $u = \dfrac{\pi}{2}$ so the curve can be represented as

$$\mathbf{r}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

Using the Theorem 1, we have

$$X = \left(s - \frac{\pi}{\sqrt{2}}\right) - \frac{1}{2^2}\cdot\frac{1}{6}\left(s - \frac{\pi}{\sqrt{2}}\right)^3$$

$$Y = \frac{1}{2\cdot 2}\left(s - \frac{\pi}{\sqrt{2}}\right)^2$$

$$Z = \frac{1}{4}\cdot\frac{1}{6}\left(s - \frac{\pi}{\sqrt{2}}\right)^3$$

Since we know κ and τ, we can find the projections of the curve on the planes using (a), (b) and (c) of (iii).

Theorem 2. The length of the common perpendicular d between the tangents at two neighbouring points with the arcual distance s between them is approximately $d = \dfrac{\kappa\tau s^3}{12}$.

Neglecting the powers of s^4, we have

$$Z \sim \frac{\kappa\tau}{6}s^3 \text{ and } XY \sim \frac{\kappa}{2}s^3$$

Eliminating s^3, we obtain $Z \sim \frac{\tau}{3}XY$ so that $\tau \sim \frac{3Z}{XY}$

Note. The above formula for κ resembles Newton's formula for curvature of plane curves.

(ii) The chord $PQ = (X^2 + Y^2 + Z^2)^{1/2} \sim s\left(1 - \frac{\kappa^2 s^2}{24}\right)$

Proof. Neglecting the powers of s^4, we get the coordinates as

$$X \sim s - \frac{\kappa^2 s^3}{6},\ Y \sim \frac{\kappa}{2}s^2 + \frac{\kappa' s^3}{6},\ Z = \frac{\kappa\tau}{6}s^3 \text{ as } s \to 0$$

Hence $X^2 + Y^2 + Z^2 = \left(s - \frac{\kappa^2 s^3}{6}\right)^2 + \left(\frac{\kappa}{2}s^2 + \frac{\kappa' s^3}{6}\right)^2 + \frac{\kappa^2\tau^2}{6^2}s^6$

Neglecting the terms of degree ≥ 5,

$$(X^2 + Y^2 + Z^2) = s^2 - \frac{1}{12}\kappa^2 s^4 = s^2\left(1 - \frac{1}{12}\kappa^2 s^2\right)$$

Thus $$\left(X^2 + Y^2 + Z^2\right)^{1/2} = s\left(1 - \frac{1}{12}\kappa^2 s^2\right)^{1/2}$$

Using the Binomial expansion on the right hand side,

$\left(X^2 + Y^2 + Z^2\right)^{1/2} \sim s\left(1 - \frac{1}{24}\kappa^2 s^2\right)$ which shows that when $\kappa \neq 0$, the arc length PQ differs from the chord PQ by a term of the third order in s.

(iii) We obtain the approximate equations of the projections of the curve on three planes. In each case we obtain the lowest power of s and eliminate s.

(a) The projection of the curve on the osculating plane is $Y = \frac{\kappa}{2}X^2,\ Z = 0$

From the equations, we obtain as in (i) $X \sim s$ and $Y \sim \frac{\kappa}{2}s^2$. Eliminating s between the two equations $Y = \frac{\kappa}{2}X^2$.

(b) The projection of the curve on the rectifying plane is

$$Z = \frac{\kappa\tau}{6}X^3,\ Y = 0$$

Retaining the terms of order 4, we get

$$\mathbf{r}(s)\cdot[\mathbf{r}'(s)\times\mathbf{r}'(0)] = \frac{\kappa^2\tau}{4}s^4 - \frac{1}{6}s^4\kappa^2\tau = \frac{1}{12}\kappa^2\tau s^4$$

Then $$d = \frac{\mathbf{r}(s)\cdot[\mathbf{r}(s)\times\mathbf{r}'(0)]}{|\mathbf{r}(s)\times\mathbf{r}'(0)|} = \frac{\kappa^2\tau s^4}{12}\cdot\frac{1}{\kappa s}\left(1-\frac{\kappa' s}{2\kappa}+\dots\right)$$

which gives $d = \dfrac{\kappa\tau s^3}{12}$ approximately.

Example 2. If O and P are the adjacent points on the curve having the arc-length $OP = s$, find the angle between the shortest distance of the tangents at O and P and the binormal at O.

From the theorem, we have

$$\mathbf{r}'(s)\times\mathbf{r}'(0) = \frac{\tau\kappa s^2}{2}\mathbf{n} - \left(s\kappa + \frac{s^2\kappa'}{2}\right)\mathbf{b} \qquad \dots(1)$$

Let θ be the angle that the shortest distance of the tangents at O and P make with the binomial.

Then $\sin\theta = \dfrac{|\mathbf{b}\times(\mathbf{r}'(s)\times\mathbf{r}'(0))|}{|\mathbf{r}'(s)\times\mathbf{r}'(0)|}$

Using (1), $|\mathbf{b}\times(\mathbf{r}'(s)\times\mathbf{r}'(0))| = \left|-\dfrac{\tau\kappa s^2}{2}\mathbf{t}\right|$ and $|\mathbf{r}'(s)\times\mathbf{r}'(0)| = \kappa s$

Hence $\sin\theta = \dfrac{|-\tau\kappa s^2|}{2\cdot\kappa s}|\mathbf{t}| = \dfrac{s}{2}\tau.$

Since θ is small, we take $\sin\theta \sim \theta$ so that $\theta \sim \dfrac{s}{2}\tau$.

1.9 THE CURVATURE AND TORSION OF A CURVE AS THE INTERSECTION OF TWO SURFACES

When the curve is defined by the parametric representation, we can find κ and τ using the formula derived earlier. When the curve is given as the intersection of two surfaces, we cannot use the earlier methods, unless we find the parametric representation of the curves from the surface equations. Now we shall develop a method to find τ and κ, when the curve is given as the intersection of two surfaces directly without finding the parametric representation of the curve.

Theorem. Let the curve be the intersection of the two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$

Let $\mathbf{h} = \nabla f\times\nabla g$. If Δ is the operator $\Delta\mathbf{r} = \mathbf{h}$, then

$$\kappa = \frac{|H|}{|\mathbf{h}|^3} \text{ and } \tau = \frac{-\Delta\mathbf{h}\cdot\Delta H}{|H|^2} \quad \text{where } H = \mathbf{h}\times\Delta\mathbf{h}.$$

Proof. Let $\mathbf{r} = \mathbf{r}(s)$ be the position vector of a point on the curve $f(x, y, z) = 0$ and $g(x, y, z) = 0$ ...(1)

$$\text{The } \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \text{ and } \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) \quad ...(2)$$

are the outward drawn normals to the surfaces (1).

Since the unit tangent at P to the curve of intersection of the two surfaces is perpendicular to both the vectors in (2), it is parallel to $\nabla f\times\nabla g$. Hence $\nabla f\times\nabla g$ is a scalar multiple of $\mathbf{t}$. Thus we have

$$\lambda\mathbf{t} = \lambda\mathbf{r}'(s) = \nabla f\times\nabla g = \mathbf{h} \text{ (say)} \quad ...(3)$$

Let us assume $\mathbf{h} = (h_1, h_2, h_3)$. Further $|\mathbf{h}| = \lambda$.

In the above equation (3), one should note that the left hand side is given in terms of the dash derivatives, whereas the right hand side is given in terms of partial derivatives. Hence let us find the relation between these two.

$$\lambda\frac{d\mathbf{r}}{ds} = \lambda\left[\frac{\partial\mathbf{r}}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial\mathbf{r}}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial\mathbf{r}}{\partial z}\cdot\frac{dz}{ds}\right]$$

$$= \lambda\left[x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right]\mathbf{r} = (h_1, h_2, h_3)$$

Since $\frac{\partial\mathbf{r}}{\partial x} = (1, 0, 0)$, $\frac{\partial\mathbf{r}}{\partial y} = (0, 1, 0)$ and $\frac{\partial\mathbf{r}}{\partial z} = (0, 0, 1)$, we obtain from the above equation, $\lambda x' = h_1$, $\lambda y' = h_2$, $\lambda z' = h_3$

Thus we have the relation

$$(\lambda x', \lambda y', \lambda z') = (h_1, h_2, h_3) \quad ...(4)$$

Let Δ be the operator defined by

$$\Delta = \lambda\frac{d}{ds} = h_1\frac{\partial}{\partial x} + h_2\frac{\partial}{\partial y} + h_3\frac{\partial}{\partial z} \quad ...(5)$$

Hence by the definition of the operator $\Delta\mathbf{r} = \mathbf{h}$ ...(6)

Operating on both sides of (3) with Δ in (5), we get

$\lambda\frac{d}{ds}(\lambda\mathbf{t}) = \Delta\mathbf{h}$ which gives

$\lambda^2\frac{d\mathbf{t}}{ds} + \lambda\lambda'\mathbf{t} = \Delta\mathbf{h}$. That is $\lambda^2\kappa\mathbf{n} + \lambda\lambda'\mathbf{t} = \Delta\mathbf{h}$ ...(7)

Taking cross product of (3) and (7), $\lambda^3\kappa\mathbf{b} = \mathbf{h}\times\Delta\mathbf{h} = H\text{(say)}$ ...(8)

Further taking dot product of (8) with itself, we get

$$\lambda^3\kappa = |H| \quad \text{or } \kappa = \frac{|H|}{\lambda^3} = \frac{|H|}{|\mathbf{h}|^3}$$

Operating on both sides of (8) with $\lambda\frac{d}{ds} = \Delta$, we get

$$\lambda\frac{d}{ds}(\lambda^3\kappa\mathbf{b}) = \Delta H \text{ giving}$$

$$-\lambda^4\kappa\tau\mathbf{n} + \lambda\mathbf{b}\frac{d}{ds}(\lambda^3\kappa) = \Delta H \qquad ...(9)$$

Taking the scalar product of (7) and (9), $-\lambda^6\kappa^2\tau = \Delta\mathbf{h}\cdot\Delta H$ which gives

$\tau = \dfrac{-\Delta\mathbf{h}\cdot\Delta H}{\lambda^6\kappa^2} = \dfrac{-\Delta\mathbf{h}\cdot\Delta H}{|H|^2}$ which completes the proof of the theorem.

Example. Find the curvature and torsion of the curve of intersection of the quadratic surfaces

$$ax^2 + by^2 + cz^2 = 1,\ a'x^2 + b'y^2 + c'z^2 = 1$$

We shall find $\mathbf{h}$, $\Delta\mathbf{h}$, $H = \mathbf{h} \times \Delta\mathbf{h}$ and $\Delta\mathbf{h}\cdot\Delta H$ and apply the formula in the theorem.

Let $$f = \frac{1}{2}[ax^2 + by^2 + cz^2 - 1],\ g = \frac{1}{2}[a'x^2 + b'y^2 + c'z^2 - 1]$$

Then $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (ax, by, cz)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = (a'x, b'y, c'z)$$

Now $$\nabla f \times \nabla g = ((bc' - b'c)yz, (ca' - c'a)zx, (ab' - a'b)xy)$$

Denoting $bc' - b'c = A$, $ca' - c'a = B$, $ab' - a'b = C$, we have

$$\nabla f \times \nabla g = xyz\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$$

Since $\nabla f \times \nabla g$ is parallel to $\mathbf{t} = \frac{d\mathbf{r}}{ds}$, we take

$$\lambda\mathbf{t} = \lambda\frac{d\mathbf{r}}{ds} = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) = \mathbf{h} \qquad ...(2)$$

Thus $$h_1 = \frac{A}{x},\ h_2 = \frac{B}{y},\ h_3 = \frac{C}{z} \text{ and } \lambda^2 = \sum\left(\frac{A}{x}\right)^2$$

Hence $$\Delta = \left(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z}\right) = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)$$

So $$\Delta \mathbf{h} = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$$

$$= -\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right) \qquad \ldots(3)$$

Hence $$H = \mathbf{h} \times \Delta \mathbf{h} = \left[\frac{BC}{y^3 z^3}(Bz^2 - Cy^2), \frac{AC}{z^3 x^3}(Cx^2 - Az^2), \frac{AB}{z^3 x^3}(Bx^2 - Ay^2)\right] \qquad \ldots(4)$$

Let us simplify $Bz^2 - Cy^2$

$$\begin{aligned} Bz^2 - Cy^2 &= (ca' - c'a)\,z^2 - (ab' - a'b)\,y^2 \\ &= a'(cz^2 + by^2) - a(c'z^2 + b'y^2) \\ &= a'(1 - ax^2) - a(1 - a'x^2) = (a' - a) \end{aligned}$$

Thus we have $Bz^2 - Cy^2 = (a' - a)$

In a similar manner, we have

$$Cx^2 - Az^2 = (b' - b) \text{ and } Ay^2 - Bx^2 = (c' - c)$$

Thus $$H = \mathbf{h} \times \Delta \mathbf{h} = \left(\frac{BC(a' - a)}{y^3 z^3}, \frac{CA(b' - b)}{z^3 x^3}, \frac{AB}{x^3 y^3}(c' - c)\right)$$

Hence $$|\mathbf{h} \times \Delta \mathbf{h}|^2 = \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \sum \frac{x^6}{A^2}(a' - a)^2$$

So $$\kappa^2 = \frac{|H|^2}{|h|^6} = \frac{1}{\left(\sum \frac{A^2}{x^2}\right)^3} \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \sum \frac{x^6}{A^2}(a' - a)^2.$$

To find τ, let us find ΔH

From equation (8) of theorem and using (4),

$$\lambda^3 \kappa \mathbf{b} = H = \frac{ABC}{x^3 y^3 z^3}\left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{Z^3}{C}(c' - c)\right)$$

That is $$\frac{x^3 y^3 z^3}{ABC} \lambda^3 \kappa \mathbf{b} = \left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c)\right) \qquad \ldots(5)$$

Let us denote by $\mu = \dfrac{x^3y^3z^3}{ABC}\lambda^3\kappa$.

Operating with $\lambda\dfrac{d}{ds} = \Delta$ on both sides of (5), we have

$$\lambda\frac{d}{ds}[\mu\mathbf{b}] = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{c}(c' - c)\right)$$

So $\lambda\mu'\mathbf{b} + \lambda\mu(-\tau\mathbf{n}) = (3x(a' - a), 3y(b' - b), cz(c' - c))$ ...(6)

From equation (7) of the theorem, we have

$$\lambda^2\kappa\mathbf{n} + \lambda\lambda'\mathbf{t} = \Delta\mathbf{h} = -\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right) \qquad ...(7)$$

Taking scalar product of (6) and (7), we obtain

$$\lambda^3\kappa\tau\mu = 3\sum\frac{A^2}{x^2}(a' - a)$$

Substituting the value of μ and simplifying, we get.

$$\lambda^6\kappa^2\tau = \frac{3ABC}{x^3y^3z^3}\sum\frac{A^2}{x^2}(a' - a)$$

Substituting the values of λ and κ, we get

$$\tau = \frac{3x^3y^3z^3}{ABC}\,\frac{\sum\frac{A^2}{x^2}(a' - a)}{\sum\frac{x^6}{A^2}(a' - a)^2}$$