APPENDIX A

# VARIATIONAL FORMULATION FOR A DISSIPATIVE SYSTEM

It may be noted that it is possible to introduce a formalism (Morse and Feshbach [7]) which will enable us to carry on calculations for dissipative systems, as though they constitute conservative systems. The method is to consider simultaneously with the dissipative system (i.e., one with friction) a system with negative friction (which may be regarded as the mirror image of the original dissipative system) such that the energy drained from the dissipative system is absorbed by the mirror image system. Thus the total energy is conserved and we can define a Lagrange function.

Consider the one-dimensional oscillator with friction, governed by the equation of motion

$$m\ddot{x} + R\dot{x} + Kx = 0.$$

Using the variational technique, we wish to derive this equation from a suitable Lagrange function. To this end, we construct the formal expression as

$$L = m\dot{x}\dot{x}^* - \frac{1}{2}R(x^*\dot{x} - x\dot{x}^*) - Kxx^*,$$

which may be regarded as the Lagrange function in the variables $x$ and $x^*$. The variable $x^*$ refers to the mirror-image oscillator with negative friction. The above $L$ gives the two momenta as

$$p = m\dot{x}^* - \frac{1}{2}Rx^*, \quad p^* = m\dot{x} + \frac{1}{2}Rx,$$

which have very little to do with the actual momentum of the oscillator. Lagrange's equations for the two systems are

$$m\ddot{x}^* - R\dot{x}^* + Kx^* = 0, \quad m\ddot{x} + R\dot{x} + Kx = 0,$$

where the equation for $x$ is precisely the one with which we started. The other equation involves the negative frictional term $-R\dot{x}^*$.

The Hamiltonian is

$$H = p\dot{x} + p^*\dot{x}^* - L = \frac{1}{m}\left(p + \frac{1}{2}Rx^*\right)\left(p^* - \frac{1}{2}Rx\right) + Kxx^*.$$

Since $x^*$ increases in amplitude as fast as $x$ decreases, $H$ will remain constant.

Although this technique is not very satisfactory even if an alternative method of solution is known, it will still be necessary for studying dissipative system, involving equation of the diffusion type given by

$$\nabla^2\Psi = a^2\frac{\partial\Psi}{\partial t}.$$

Here $\Psi$ is the density of a diffusing fluid and $a^2$ is the diffusion coefficient (assumed constant). Let $\Psi^*$ refer to the density of the mirror-image system. We construct the Lagrange function $L$ as

$$L = -(\nabla\Psi \cdot \nabla\Psi^*) - \frac{a^2}{2}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right)$$

so that the canonical momentum densities are

$$p = \frac{\partial L}{\partial \Psi} = -\frac{1}{2}a^2\Psi^*, \quad p^* = \frac{1}{2}a^2\Psi.$$

The Euler equations corresponding to $L$ above are

$$\nabla^2\Psi = a^2\frac{\partial \Psi}{\partial t}, \quad \nabla^2\Psi^* = -a^2\frac{\partial \Psi^*}{\partial t},$$

where the first equation is the diffusion equation with which we started.

## APPENDIX B

# EKELAND'S VARIATIONAL PRINCIPLE

We have already seen in Section 1.11 that a functional bounded below attains its infimum if it has some type of continuity in a topology that renders compactness, to the domain of the said functional. However, in some situations of interest in applications, this is not the case. Take, for instance, the functionals (defined in an infinite-dimensional Hilbert space) which are continuous in the norm topology, but not in the weak topology. Such problems can be tackled by Ekeland variational principle (Figueiredo [8]). This principle has found a variety of applications in different fields of analysis. We state Ekeland's principle (weak form) as follows: Let $(X, d)$ be a complete metric space. Let further $\Phi : X \to RU \{+\infty\}$ be lower semi-continuous and bounded below. Then for a given $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that

$$\Phi(u_\varepsilon) \le inf_X \ \Phi + \varepsilon$$

$$\Phi(u_\varepsilon) < \Phi(u) + \varepsilon d(u, u_\varepsilon)$$

for every $u \in X$ with $u \neq u_\varepsilon$.

A stronger version of Ekeland's principle is as follows: Let $(X, d)$ be a complete metric space and $\Phi : X \to RU \{+\infty\}$ be a lower semi-continuous function which is bounded below.

Let $\varepsilon > 0$ and $\bar{u} \in x$ be given so that

$$\Phi(\bar{u}) \le inf_X \Phi + \frac{\varepsilon}{2}.$$

Then for a given $\lambda > 0$, there exists $u_\lambda \in X$, such that

(i) $\Phi(u_\lambda) \le \Phi(\bar{u})$, CHAPTER 2

# VARIATIONAL PROBLEMS WITH MOVING BOUNDARIES

## 2.1 Functional of the Form $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$

In the problems discussed so far, we have taken the boundary points $(x_1, y_1)$ and $(x_2, y_2)$ in the functional

$$I\,[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx \tag{2.1}$$

as fixed. Let us now consider the case when one or both the boundary points can move. This means that the class of admissible curves is extended because, in addition to the comparison curves with fixed boundary points, we have to admit curves with variable boundary points.

It is clear that if on a curve $y = y(x)$, an extremum is attained in a problem with moving boundary points, then surely enough the extremum is all the more attained on a restricted class of curves with common (fixed) boundary points. Thus the curves $y = y(x)$ on which extremum of the above functional is attained in a moving boundary problem must be solutions of the Euler equation

$$F_y - \frac{d}{dx} F_{y'} = 0$$

so that these curves must be extremals.

In the problem with fixed boundary points, the two constants in the solution of Euler's equation are determined from the two boundary conditions at the fixed points at $(x_1, y_1)$ and $(x_2, y_2)$. But in a moving boundary problem, one or both of these conditions are missing, and the arbitrary constants in the general solution of Euler's equation have to be obtained from the vanishing of the variation $\delta I$, which is the necessary condition for extremum.

For the sake of simplicity, let us assume that one of the boundary points $(x_1, y_1)$ is fixed while the other boundary point $(x_2, y_2)$ can move and pass to the point $(x_2 + \delta x_2, y_2 + \delta y_2)$. Since, as shown above, the extremum in a moving boundary problem is attained only on extremals, i.e., on solutions $y = y(x, C_1, C_2)$ of Euler's equations, from now on we consider the values of the functional $I$ on such curves. Thus, $I\,[y(x, C_1, C_2)]$ reduces to a function of the parameters $C_1$ and $C_2$ and of the limits of integration. We shall call the admissible curves $y = y(x)$ and $y = y(x) + \delta y(x)$ close if $|\delta y|$ and $|\delta y'|$ are also small.

The extremals passing through $(x_1, y_1)$, form a pencil of extremals $y = y(x, C_1)$. The functional $I\,[y(x, C_1)]$ on the curves of this pencil becomes a function of $C_1$ and $x_2$. If the curves of the pencil $y = y(x, C_1)$ do not intersect in the neighbourhood of the extremal (see Fig. 2.1), then $I\,[y(x, C_1)]$ may be considered as a single-valued function of $(x_2, y_2)$. This is because the specification of $(x_2, y_2)$ determines the extremal of the pencil uniquely, and hence determines the value of the functional.

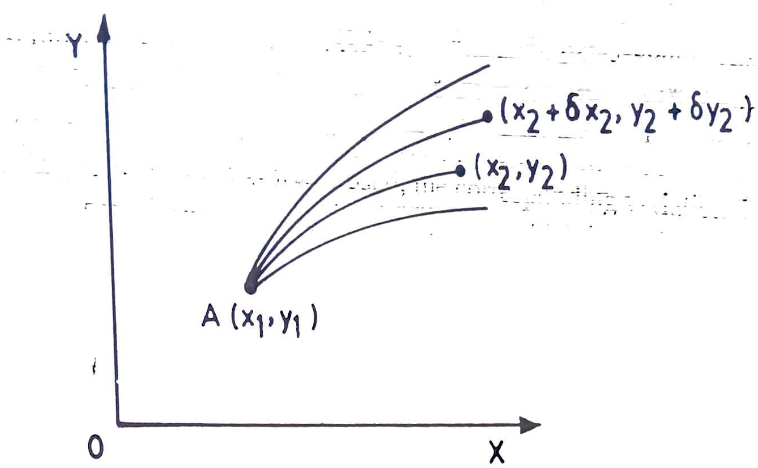


**Fig. 2.1** A pencil of extremals through a given point.

Let us determine the variation of the functional $I\,[y(x, C_1)]$ when the boundary point moves from $(x_2, y_2)$ to $(x_2 + \delta x_2, y_2 + \delta y_2)$. Thus the increment $\Delta I$ is given by

$$\begin{aligned}\Delta I &= \int_{x_1}^{x_2+\delta x_2} F(x, y+\delta y, y'+\delta y')\,dx - \int_{x_1}^{x_2} F(x, y, y')\,dx \\ &= \int_{x_1}^{x_2} [F(x, y+\delta y, y'+\delta y') - F(x, y, y')]\,dx \\ &\quad + \int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, y'+\delta y')\,dx. \end{aligned} \tag{2.2}$$

Using the mean value theorem, the second term on the right-hand side of (2.2) can be written as

$$\int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, y'+\delta y')\,dx = [F]_{x_2+\theta\cdot\delta x_2}\cdot\delta x_2 \tag{2.3}$$

where $0 < \theta < 1$. But by virtue of the continuity of $F$, we may write

$$F|_{x_2+\theta\cdot\delta x_2} = F|_{x_2} + \varepsilon \tag{2.4}$$

where $\varepsilon$ is an infinitesimal such that $\varepsilon \to 0$ as $\delta x_2 \to 0$ and $\delta y_2 \to 0$. Thus by (2.3) and (2.4) we have

$$\int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, y'+\delta y')\,dx = F|_{x=x_2}\cdot\delta x_2 + \varepsilon\,\delta x_2 \tag{2.5}$$

Using Taylor's theorem we now transform the first term on the right side of (2.2) as

$$\int_{x_1}^{x_2} [F(x, y + \delta y, y' + \delta y') - F(x, y, y')]\, dx$$

$$= \int_{x_1}^{x_2} [F_y(x, y, y')\, \delta y + F_{y'}(x, y, y')\, \delta y']\, dx + R$$

where $R$ is an infinitesimal of the order higher than that of $\delta y$ or $\delta y'$. Further, the linear part on the right side of the above integral can be, after integration by parts, reduced to

$$[F_{y'}\, \delta y]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left( F_y - \frac{d}{dx} F_{y'} \right) \delta y\, dx.$$

Since the values of the functional $I$ are taken only on the extremals, the integral in the second term of the above expression vanishes since $F_y - \frac{d}{dx} F_{y'} \equiv 0$ and the expression then becomes

$$(F_{y'}\, \delta y)_{x=x_2}$$

since $(\delta y)_{x_1} = 0$. Note that $(\delta y)_{x_2}$ is not equal to $\delta y_2$ since $\delta y_2$ is the increment of $y_2$ when the boundary point is displaced to $(x_2 + \delta x_2, y_2 + \delta y_2)$. But $(\delta y)_{x_2}$ is the increment of the ordinate at the point $x_2$ when passing from the extremal joining $(x_1, y_1)$ and $(x_2, y_2)$ to the one joining $(x_1, y_1)$ and $(x_2 + \delta x_2, y_2 + \delta y_2)$ as shown in Fig. 2.2.

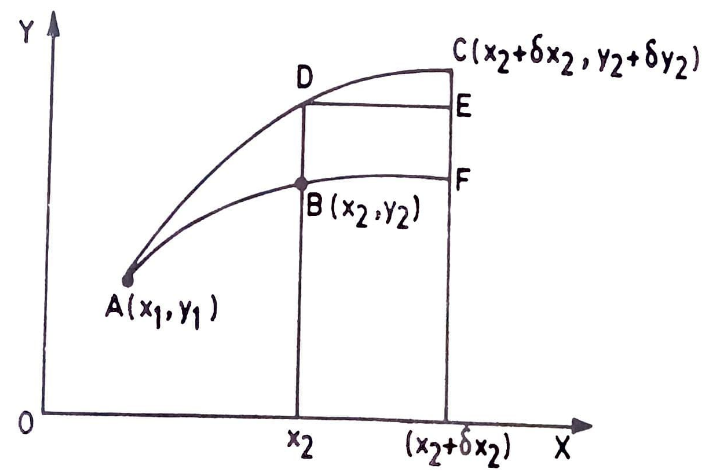


Fig. 2.2 Extremizing curve and a neighbouring curve through a given point.

It is evident from Fig. 2.2 that $BD = (\delta y)_{x_2}$ and $FC = \delta y_2$. Further, $EC = y'(x_2) \cdot \delta x_2$ and hence $BD = FC - EC$ gives

$$(\delta y)_{x_2} \approx \delta y_2 - y'(x_2) \cdot \delta x_2$$

This equality is valid within the infinitesimals of higher order. Thus

$$\int_{x_1}^{x_2} [F(x, y + \delta y, y' + \delta y') - F(x, y, y')]\, dx = F_{y'}|_{x=x_2} \cdot (\delta y_2 - y'(x_2)\, \delta x_2).$$

Finally, using (2.5) and the above relation in (2.2), we obtain

$$\delta I = F|_{x=x_2} \cdot \delta x_2 + F_{y'}|_{x=x_2} \cdot (\delta y_2 - y'(x_2)\delta x_2)$$

$$= (F - y'F_{y'})|_{x=x_2} \cdot \delta x_2 + F_{y'}|_{x=x_2} \cdot \delta y_2 \qquad (2.6)$$

By virtue of the fact that $\delta x_2$ and $\delta y_2$ are independent, the necessary condition for the extremum $\delta I = 0$ then gives

$$(F - y' F_{y'})\,|_{x=x_2} = 0, \qquad (2.7)$$

$$F_{y'}\,|_{x=x_2} = 0$$

Cases, however, arise when $\delta x_2$ and $\delta y_2$ are not independent. For example, if the boundary point $(x_2, y_2)$ moves along the curve

$$y_2 = \phi(x_2), \qquad (2.8)$$

then $\delta y_2 = \phi'(x_2) \cdot \delta x_2$. Thus from (2.6), we get Q.P 8m

$$[F + (\phi' - y')\, F_{y'}]_{x=x_2} \cdot \delta x_2 = 0.$$

[write the transversality condition for the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx$]

Since $\delta x_2$ is arbitrary, we must have

$$[F + (\phi' - y')\, F_{y'}]_{x=x_2} = 0, \qquad (2.9)$$

which provides the condition at the free boundary. This is known as the transversality condition. The conditions (2.8) and (2.9) suffice to determine the extremals of the pencil $y = y(x, C_1)$ on which an extremum may be attained. A condition similar to (2.9) obtains if the boundary point $(x_1, y_1)$ moves along another prescribed curve $y_1 = \Psi(x_1)$.

There is one simple case when the transversality condition (2.9) reduces to the orthogonality condition. Consider the case when $F$ in (2.1) is given by $A(x, y) \cdot (1 + y'^2)^{1/2}$, where $A(x, y)$ does not vanish at the movable boundary point $x_2$. In this case (2.9) reduces to

$$A(x, y) \cdot (1 + \phi' y')/(1 + y'^2)^{1/2} = 0 \text{ at } x = x_2.$$

Since $A(x, y) \neq 0$ at $x = x_2$, we have $y' = -1/\phi'$ at $x = x_2$, which is the orthogonality condition.

Q.P 5m ***Example 1.*** Find the shortest distance between the parabola $y = x^2$ and the straight line $x - y = 5$.

Y V. I ***Solution.*** The problem is to find the extremum of the functional

$$I[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx$$

subject to the condition that the left end of the extremal moves along $y = x^2$ while the right end moves along $x - y = 5$. Thus the transversality condition (2.9) gives

$$[(1 + y'^2)^{1/2} + (2x - y')y'(1 + y'^2)^{-1/2}]_{x=x_1} = 0. \tag{2.10}$$

$$[(1 + y'^2)^{1/2} + (1 - y')y'(1 + y'^2)^{-1/2}]_{x=x_2} = 0. \tag{2.11}$$

Since the general solution of Euler's equation for the above functional is $y = C_1x + C_2$ (where $C_1$ and $C_2$ are constants), it follows that $y' = C_1$. Further, both the end points lie on the extremal $y = C_1x + C_2$, hence we must have

$$C_1x_1 + C_2 = x_1^2, \tag{2.12}$$

$$C_1x_2 + C_2 = x_2 - 5. \tag{2.13}$$

Replacing $y'$ in (2.10) and (2.11) by $C_1$ and solving the resulting equations along with (2.12) and (2.13), we get

$$C_1 = -1, \quad C_2 = \frac{3}{4}, \quad x_1 = \frac{1}{2}, \quad x_2 = \frac{23}{8}.$$

Thus the required extremal is $y = -x + \frac{3}{4}$ and the shortest distance between the given parabola and the straight line is

$$L = \int_{1/2}^{23/8} (1 + 1)^{1/2}\, dx = \frac{19\sqrt{2}}{8}.$$

5 (+) (+) GP 10m V V I

*Example 2.* Using only the basic necessary condition $\delta I = 0$, find the curve on which an extremum of the functional

$$I[y(x)] = \int_0^{x_1} \frac{(1 + y'^2)^{1/2}}{y}\, dx, \quad y(0) = 0$$

can be achieved if the second boundary point $(x_1, y_1)$ can move along the circumference $(x - 9)^2 + y^2 = 9$.

*Solution.* Denoting the integrand of the functional by $F$, the extremals are clearly the integral curves

$$F - y'F_{y'} = \text{constant}$$

or Euler equation. This gives

$$1 + y'^2 = C/y^2, \qquad C = \text{constant}.$$

Its solution is the two-parameter family of circles

$$(x - C_1)^2 + y^2 = C_2^2. \tag{2.14}$$

The boundary condition $y(0) = 0$ leads to $C_1 = C_2$. Further, since the integrand is of the form $A(x, y)\,(1 + y'^2)^{1/2}$, the transversality condition at the movable boundary point $(x_1, y_1)$ reduces to the orthogonality condition. Thus the required extremal will be the arc of a circle belonging to (2.14) which is orthogonal to $(x - 9)^2 + y^2 = 9$ (see Fig. 2.3).

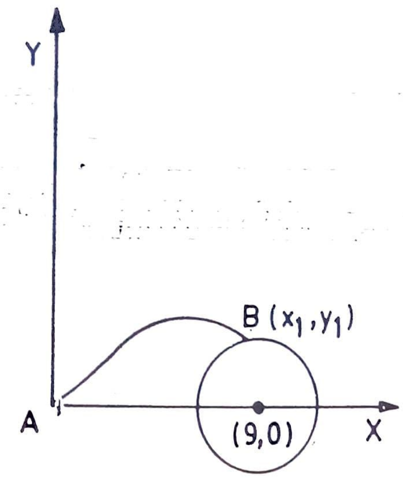


**Fig. 2.3** Extremizing curve satisfying the orthogonality condition.

Since $B(x_1, y_1)$ lies on both circles, we must have

$$x_1^2 - 18x_1 + y_1^2 = -72, \quad x_1^2 - 2C_1x_1 + y_1^2 = 0$$

which gives

$$x_1(C_1 - 9) = -36 \tag{2.15}$$

In view of orthogonality of the two circles at $(x_1, y_1)$, the tangent to (2.14) at $B$ passes through the centre (9, 0) of the given circle. This yields

$$(9 - C_1)x_1 = 9C_1. \tag{2.16}$$

Solving (2.15) and (2.16), we find that $C_1 = 4$ and $x_1 = 36/5$ so that the required extremals (2.14) are the arcs of the circles $y = \pm\,(8x - x^2)^{1/2}$.

## 2.2 Variational Problem with a Movable Boundary for a Functional Dependent on Two Functions

Consider the functional

$$I[y(x), z(x)] = \int_{x_1}^{x_2} F(x, y(x), z(x), y'(x), z'(x))\, dx. \tag{2.17}$$

Let the point $A(x_1, y_1, z_1)$ corresponding to the lower limit in the above integral be fixed, and let the other point $B(x_2, y_2, z_2)$ move in an arbitrary manner, or along a given curve or surface.

It is clear that the extremum can be attained only on the integral curves of Euler equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \qquad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

The general solution of these equations contains four arbitrary constants. Since the boundary point $A(x_1, y_1, z_1)$ is fixed, it is possible to eliminate two arbitrary constants. The other two constants have to be determined from the necessary condition $\delta I = 0$ for extremum, where $\delta I$ is the variation of $I$. This variation may be computed in precisely the same way as in Section 2.1. Hence, $\delta I = 0$ gives

$$\delta I = (F - y'F_{y'} - z'F_{z'})_{x=x_2} \cdot \delta x_2 + (F_{y'})_{x=x_2} \cdot \delta y_2 + (F_{z'})_{x=x_2} \cdot \delta z_2$$
$$= 0 \tag{2.18}$$

for an extremum.

If the variations $\delta x_2$, $\delta y_2$ and $\delta z_2$ are independent, then (2.18) gives

$$[F - y'F_{y'} - z'F_{z'}]_{x=x_2} = 0, \quad [F_{y'}]_{x=x_2} = 0, \quad [F_{z'}]_{x=x_2} = 0.$$

If the boundary point $(x_2, y_2, z_2)$ moves along some curve $y_2 = \phi(x_2)$, $z_2 = \Psi(x_2)$, then $\delta y_2 = \phi'(x_2) \cdot \delta x_2$ and $\delta z_2 = \Psi'(x_2) \cdot \delta x_2$. Thus from (2.15) we have

$$[F + (\phi' - y')F_{y'} + (\Psi' - z')F_{z'}]_{x=x_2} \cdot \delta x_2 = 0.$$

Since $\delta x_2$ is arbitrary, we have

$$[F + (\phi' - y')F_{y'} + (\Psi' - z')F_{z'}]_{x=x_2} = 0. \tag{2.19}$$

This is the transversality condition in the problem of extremum of (2.17). Along with the equations $y_2 = \phi(x_2)$, $z_2 = \Psi(x_2)$, the condition (2.19) gives the equations necessary for determining the two arbitrary constants in the general solution of Euler's equations.

If, on the other hand, the boundary point $B(x_2, y_2, z_2)$ moves along a given surface $z_2 = \phi(x_2, y_2)$, then $\delta z_2 = \phi_{x_2}\ \delta x_2 + \phi_{y_2}\ \delta y_2$ such that the variations $\delta x_2$ and $\delta y_2$ are arbitrary. In this case (2.19) reduces to

$$[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} \cdot \delta x_2 + [F_{y'} + \phi_y F_{z'}]_{x=x_2} \cdot \delta y_2 = 0.$$

Since $\delta x_2$ and $\delta y_2$ are independent, we find

$$[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} = 0$$

$$[F_{y'} + \phi_y F_{z'}]_{x=x_2} = 0. \tag{2.20}$$

These two conditions, together with $z_2 = \phi(x_2, y_2)$, enable us to determine two arbitrary constants in the general solution of Euler's equation.

It may be easily seen that for the functional

$$I = \int_{x_1}^{x_2} A(x, y, z) \bullet (1 + y'^2 + z'^2)^{1/2}\, dx$$

with the end point $(x_2, y_2, z_2)$ lying on the surface $z = \phi(x, y)$, the transversality condition reduces to the orthogonality condition of the extremal to the surface $z = \phi(x, y)$. From this, it immediately follows that the shortest distance between two surfaces $z = L(x, y)$, and $z = M(x, y)$, can only be attained on the straight lines, which are orthogonal to both these surfaces.

Q.P 9m
write the condition for variational problem with movable boundary.

The above arguments can be readily extended to a functional of the form

$$I = \int_{x_1}^{x_2} F(x, y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x))\, dx$$

such that at the moving point $x_2$, we have

$$\left(F - \sum_{i=1}^{n} y_i' F_{y_i'}\right)_{x = x_2} \cdot \delta x_2 + \sum_{i=1}^{n} (F_{y_i'})_{x = x_2} \cdot \delta y_{i2} = 0. \qquad (2.21)$$

Q.P 2xy
5m
S 10m V.V.I ($x$) $yz$ ($x$)

**Example 3.** Find the extremum of the functional

$$I = \int_{x_1}^{x_2} (y'^2 + z'^2 + \underset{2xy}{2yz})\, dx \text{ with } y(0) = 0,$$

$z(0) = 0$ and the point $(x_2, y_2, z_2)$ moves over the fixed plane $x = x_2$.

***Solution.*** The Euler equation in this case gives $z'' - y = 0$ and $y'' - z = 0$ leading to $y^{iv} - y = 0$. The solutions are

$$y = C_1 \cosh x + C_2 \sinh x + C_3 \cos x + C_4 \sin x \qquad (2.22)$$

$$z = C_1 \cosh x + C_2 \sinh x - C_3 \cos x - C_4 \sin x \qquad (2.23)$$

The conditions $y(0) = z(0) = 0$ give $C_1 = C_3 = 0$. Further, the condition at the moving boundary point $(x_2, y_2, z_2)$ can be derived from (2.18) with $\delta x_2 = 0$ (since $x_2$ is fixed) as

$$(F_{y'})_{x=x_2} = 0, \quad (F_{z'})_{x=x_2} = 0$$

giving

$$y'(x_2) = 0, \quad z'(x_2) = 0.$$

Thus (2.22) and (2.23) lead to

$$C_2 \cosh x_2 + C_4 \cos x_2 = 0, \qquad C_2 \cosh x_2 - C_4 \cos x_2 = 0.$$

If $\cos x_2 \neq 0$, then $C_2 = C_4 = 0$ and, therefore, an extremum is attained on $y = 0$, $z = 0$. But if $\cos x_2 = 0$, then $C_2 = 0$ and $C_4$ remains arbitrary. In this case the extremal is $y = C_4 \sin x$, $z = -C_4 \sin x$.

## 2.3 One-Sided Variations

Let us consider again the functional (2.1) and suppose that a restriction is imposed on the class of permissible curves in such a way that the curves cannot pass through points of a certain region $R$ bounded by the curve $\Psi(x, y) = 0$, as shown in Fig. 2.4.

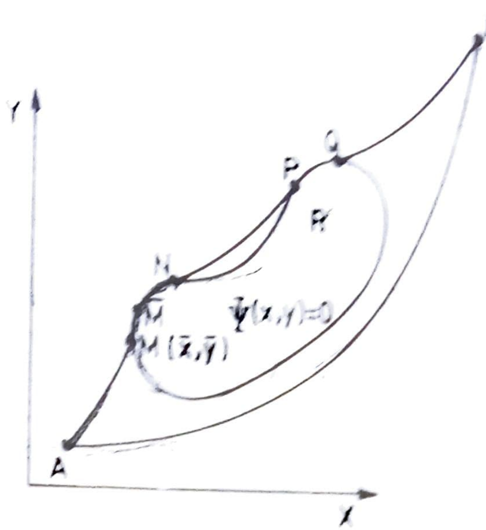


Fig. 2.4 An extremal not allowed to pass through a given region

In such a problem the extremizing curve $C$ either passes through a region which is completely outside $R$ or $C$ consists of arcs lying outside $R$ and also consists of parts of the boundary of the region $R$. In the former case the presence of $R$ does not at all influence the properties of the functional and its variation in the neighbourhood of $C$ and hence, the extremizing curve must be an extremal. In the latter case only one-sided variations of the curve $C$ are possible on parts of the boundary of the region $R$ since the permissible curves are prohibited from entering $R$. Parts of the curve $C$ outside the boundary of $R$ are clearly extremals since on these parts two-sided variations (unaffected by the region $R$) are possible. We now derive conditions at the points of transition $M$, $N$, $P$ and $Q$.

While computing the variation of the functional $\delta I$ of the functional

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx = \int_{x_1}^{\bar{x}} F(x, y, y')\,dx + \int_{\bar{x}}^{x_2} F(x, y, y')\,dx$$

$$= I_1 + I_2 \qquad (2.24)$$

we suppose that the variation is caused solely by the displacement of the point $M(\bar{x}, \bar{y})$ on the curve $\Psi(x, y) = 0$. Thus for any position $M$ on the curve we may consider that $AM$ is an extremal and the segment $MNPQB$ does not vary. For the extremal in (2.24) the upper boundary point $\bar{x}$ moves along the boundary of the region $R$ and if $y = \phi(x)$ be the equation of the boundary (as deduced from $\Psi(x, y) = 0$), then it follows from (2.9) that

$$\delta I_1 = [F + (\phi' - y')F_{y'}]_{x=\bar{x}} \cdot \delta \bar{x} \qquad (2.25)$$

The functional $I_2 = \int_{\bar{x}}^{x_2} F(x, y, y')\,dx$ also has a moving boundary point $\bar{x}$.

But in the neighbourhood of this point, the curve $y = \phi(x)$ on which an extremum can be achieved does not vary. Thus,

$$\Delta I_2 = \int_{\bar{x}+\Delta\bar{x}}^{x_2} F(x, y, y')\,dx - \int_{\bar{x}}^{x_2} F(x, y, y')\,dx$$

$$= -\int_{\bar{x}}^{\bar{x}+\Delta\bar{x}} F(x, y, y')\,dx = -\int_{\bar{x}}^{\bar{x}+\Delta\bar{x}} F(x, \phi(x), \phi'(x))\,dx,$$

since $y = \phi(x)$ on the interval $(\bar{x}, \bar{x} + \Delta\bar{x})$. Using the mean value theorem and the continuity of $F$, we get

$$\Delta I_2 = -[F(x, \phi, \phi')]_{x=\bar{x}} \cdot \Delta\bar{x} + \alpha\Delta\bar{x},$$

where $\alpha \to 0$ as $\Delta\bar{x} \to 0$. This gives

$$\delta I_2 = -[F(x, \phi, \phi')]_{x=\bar{x}} \cdot \Delta\bar{x}. \tag{2.26}$$

Combining (2.25) and (2.26), we find that

$$\delta I = [F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')]_{x=\bar{x}} \cdot \delta\bar{x}$$

with $y(\bar{x}) = \phi(\bar{x})$.

Since $\delta\bar{x}$ is arbitrary, it follows that the necessary condition $\delta I = 0$ for an extremum reduces to

$F_{y'}(x,y,y') = \dfrac{F(x,y,y') - F(\ldots}{(y' - \phi')}$

$$[F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')]_{x=\bar{x}} = 0.$$

Applying the mean value theorem to this equation, we get

$$[(y' - \phi')\{F_{y'}(x, y, q) - F_{y'}(x, y, y')\}]_{x=\bar{x}} = 0. \tag{2.27}$$

where $q$ lies between $y'(\bar{x})$ and $\phi'(\bar{x})$.

Applying the mean value theorem once more to (2.27), we finally obtain

$$[(y' - \phi')(q - y')F_{y'y'}(x, y, \bar{q})]_{x=\bar{x}} = 0$$

where $\bar{q}$ lies between $q$ and $y'(\bar{x})$.

Assume $F_{y'y'}(x, y, \bar{q}) \neq 0$ (which is a valid assumption for many variational problems). In this case $y'(\bar{x}) = \phi'(\bar{x})$ because $q = y'$ only when $y'(\bar{x}) = \phi'(\bar{x})$.

Hence we conclude that at the point $M$, the extremal $AM$ meets the boundary curve $MN$ tangentially.

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**Example 4.** Find the shortest path from the point $A(-2, 3)$ to the point $B(2, 3)$ located in the region $y \leq x^2$.

***Solution.*** The problem is to find the extremum of the functional

$$I[y] = \int_{-2}^{2} [1 + y'^2(x)]^{1/2}\,dx$$

subject to the conditions

$$y \leq x^2, \quad y(-2) = 3, \quad y(2) = 3.$$

Clearly, the extremals of $I[y]$ are the straight lines $y = C_1 + C_2x$.

If $F$ is the integrand in $I[y]$, then $F_{y'y'}$ $(= [1 + y'^2(x)]^{3/2}) \neq 0$. Hence by the result of Section 2.3, the desired extremal will consist of portions of the straight lines $AP$ and $QB$ both tangent to the paraboa $y = x^2$ and of the portion $POQ$ of the parabola (Fig. 2.5).

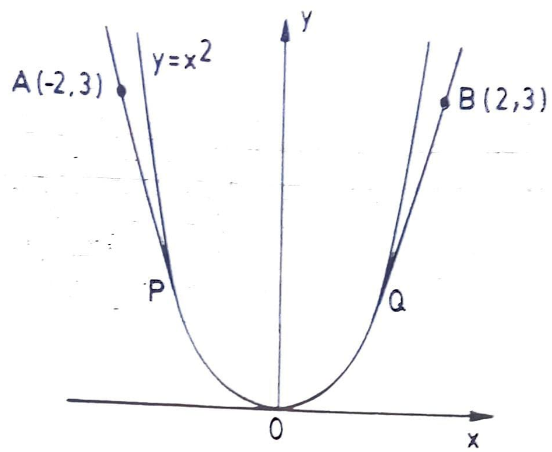


**Fig. 2.5** Extremal through two given points outside a parabolic region.

Let the abscissae of $P$ and $Q$ be $-\bar{x}$ and $\bar{x}$, respectively. Then the condition of tangency of $AP$ and $BQ$ at $P$ and $Q$ demands

$$C_1 + C_2\bar{x} = \bar{x}^2, \quad C_2 = 2\bar{x} \tag{2.28}$$

Since the tangent $QB$ passes through (2, 3),

$$C_1 + 2C_2 = 3. \tag{2.29}$$

Solution of (2.28) and (2.29) gives two values for $\bar{x}$ viz., $\bar{x}_1 = 1$ and $\bar{x}_2 = 3$. The second value is clearly inadmissible and so $\bar{x}_1 = 1$. This gives from (2.28) $C_2 = 2$. Thus the required extremal is

$$y = \begin{cases} -2x - 1 & \text{if } -2 \leq x \leq -1, \\ x^2 & \text{if } -1 \leq x \leq 1, \\ 2x - 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

This obviously minimizes the functional.

## 2.4 Reflection and Refraction of Extremals

So far we have considered variational problems in which the desired function $y = y(x)$ is continuous with continuous derivatives. However, there do arise problems, where an extremum is achieved on curves (extremals), having corner points. Take for instance the problems involving reflection and refraction of extremals, which might be regarded as the generalization of the problems of reflection and refraction of light.

Let us begin by stating the problem of reflection of extremals: find the curve that extremizes the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

and passes through $A(x_1, y_1)$ and $B(x_2, y_2)$, the curve must arrive at $B$ only after reflection from a given curve $y = f(x)$ (Fig. 2.6).

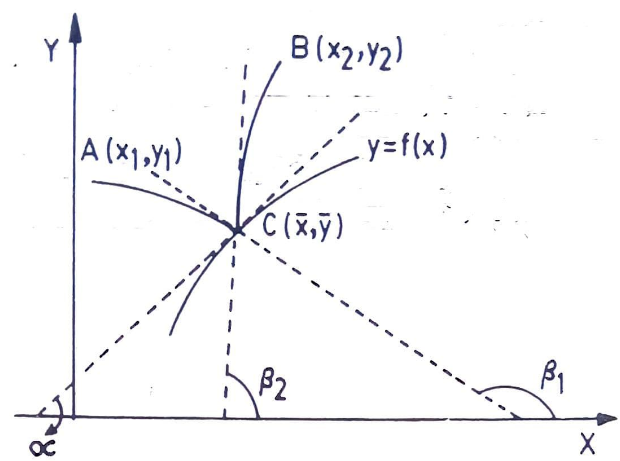


**Fig. 2.6** Extremal through two given points after reflection from a given curve.

It is conceivable that at the point of reflection, $C(\bar{x}, \bar{y})$, the desired extremal may have a corner point so that $y'(\bar{x} - 0)$ and $y'(\bar{x} + 0)$ are, in general, distinct. Thus the functional $I$ can be written as

$$I[y(x)] = \int_{x_1}^{\bar{x}} F(x, y, y')\,dx + \int_{\bar{x}}^{x_2} F(x, y, y')\,dx$$

so that the necessary condition for an extremum given by $\delta I = 0$ reduces to

$$\delta \int_{x_1}^{\bar{x}} F(x, y, y')\,dx + \delta \int_{\bar{x}}^{x_2} F(x, y, y')\,dx = 0. \tag{2.30}$$

Evidently the curves $AC$ and $CB$ are both extremals because, if we assume that one of the curves is already found, and we vary the other one alone, then the problem reduces to finding the extremal of $\int_{x_1}^{\bar{x}} F\,dx \left( \text{or} \int_{\bar{x}}^{x_2} F\,dx \right)$ with fixed boundary points. Thus for calculating the variation of the functionals in (2.30), we will assume that the functionals are considered only on extremals with corner point $C$. Thus

$$\delta \int_{x_1}^{\bar{x}} F(x, y, y')\,dx = [F + (f' - y')F_{y'}]_{x=\bar{x}-0} \cdot \delta x_1$$

$$\delta \int_{\bar{x}}^{x_2} F(x, y, y')\,dx = -[F + (f' - y')F_{y'}]_{x=\bar{x}+0} \cdot \delta x_1.$$

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Substituting the above expressions in (2.30) and remembering that $\delta x_1$ is arbitrary, we find that

$$[F + (f' - y')F_{y'}]_{x=x_1-0} = [F + (f' - y')F_{y'}]_{x=x_1+0} \tag{2.31}$$

This relation assumes a particularly simple form for a functional of the form

$$I[y(x)] = \int_{x_1}^{x_2} A(x, y) \cdot (1 + y'^2)^{1/2}\, dx$$

In this case (2.31) becomes

$$A(x_1, y_1) \cdot \left[\sqrt{1 + y'^2} + \frac{(f' - y')y'}{\sqrt{1 + y'^2}}\right]_{x=x_1-0}$$

$$= A(x_1, y_1)\left[\sqrt{1 + y'^2} + \frac{(f' - y')y'}{\sqrt{1 + y'^2}}\right]_{x=x_1+0}$$

Assuming $A(x_1, y_1) \neq 0$, the above relation gives

$$[(1 + f'y')/\sqrt{1 + y'^2}]_{x=x_1-0} = [(1 + f'y')/\sqrt{1 + y'^2}]_{x=x_1+0}$$

Referring to Fig. 2.6, we find that $y'(\bar{x} - 0) = \tan \beta_1$, $y'(\bar{x} + 0) = \tan \beta_2$, $f'(\bar{x}) = \tan \alpha$.

Thus the above reflection condition becomes

$$-(1 + \tan \alpha \cdot \tan \beta_1)/\sec \beta_1 = (1 + \tan \alpha \cdot \tan \beta_2)/\sec \beta_2$$

where the negative sign on the left arises because $\sec \beta_1 < 0$. On simplification the above equation gives

$$-\cos(\alpha - \beta_1) = \cos(\alpha - \beta_2)$$

which expresses the equality of the angle of incidence, and the angle of reflection.

As for the refraction of extremals, we consider the specific problem of refraction of light as follows. The velocity of light in medium 1 is $v_1$ (constant) while in medium 2, it is $v_2$ (constant). Medium 1 is separated from medium 2 by the curve $y = \phi(x)$. We wish to derive the law of refraction of a light ray traversing from a point $A$ in medium 1 to a point $B$ in medium 2 subject to the condition that the light ray traverses this path in the shortest time interval in accordance with Fermat's principle.

Clearly, the problem reduces to finding the minimum of the functional

$$I[y] = \int_a^c \frac{\sqrt{1 + y'^2}}{v_1}\, dx + \int_c^b \frac{\sqrt{1 + y'^2}}{v_2}\, dx$$

$$= \int_a^c F_1(y')\, dx + \int_c^b F_2(y')\, dx \tag{2.32}$$

where $a$ and $b$ are the abscissae of $A$ and $B$ and $c$ is the abscissa of the point $C$, where the extremal from $A$ to $B$ meets the curve $y = \phi(x)$. It is conceivable that

K. Backiyalakshmi



the extremum suffers a jump in slope at the point $C$. The necessary condition $\delta I = 0$ for an extremum, then becomes as in the case of reflection of extremals

$$[F_1 + (\phi' - y')F_{1y'}]_{x=c-0} = [F_2 + (\phi' - y')F_{2y'}]_{x=c+0} \tag{2.33}$$

Since in the present problem $F_1$ and $F_2$ given by (2.32) are functions of $y'$ only, the extremals are straight lines such that $y = y_1(x) = lx + m$ is the extremal in medium 1 and $y = y_2(x) = px + q$ is the extremal in medium 2, ($l, m, p$ and $q$ being constants). Substituting for $F_1$ and $F_2$ from (2.32) in (2.33), we get

$$(1 + \phi' y_1')/v_1 \cdot \sqrt{1 + y_1'^2} = (1 + \phi' y_2')/v_2 \cdot \sqrt{1 + y_2'^2} \tag{2.34}$$

Let $\gamma$ be the angle formed with the $x$-axis by the tangent to the boundary curve $y = \phi(x)$ at the point $C$, $\alpha$, the angle of the left hand ray with the $x$-axis, and $\beta$, the angle of the right hand ray with the $x$-axis. Then $\phi' = \tan\gamma$, $y_1' = \tan\alpha$, $y_2' = \tan\beta$ so that the condition (2.34) reduces to

$$\frac{\cos(\gamma - \alpha)}{v_1} = \frac{\cos(\gamma - \beta)}{v_2}$$

where $\gamma - \alpha$ and $\gamma - \beta$ are the angles between the rays and the tangent to the boundary curve. Introducing in their place the angles $\theta$ and $\phi$ between the normal to the boundary curve, and the incident and refracted rays, we obtain the well known law of refraction of light (Snell's law) as

$$\frac{\sin\theta}{\sin\phi} = \frac{v_1}{v_2} = \text{constant.}$$

From the foregoing discussion it should not, however, be thought that a corner point (where there is a discontinuity in the slope) on an extremal occurs only in problems involving reflection and refraction of extremals. Such a point can, in fact, exist on an extremal of the functional

$$\int_{x_1}^{x_2} F(x, y, y')\,dx$$

where $F$ is three times differentiable and the admissible curves pass through the given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let us now determine the conditions that must be satisfied by the solutions of the extremum problems of the above functional. For simplicity, we assume that the extremal for this problem has just one corner point at $C(\bar{x}, \bar{y})$ as shown in Fig. 2.7. Now $AC$ and $CB$ are integral curves of Euler equation for the functional. If $C$ moves in any fashion, we get from

$$I[y] = \int_{x_1}^{\bar{x}} F(x, y, y')\,dx + \int_{\bar{x}}^{x_2} F(x, y, y')\,dx,$$

the necessary condition $\delta I = 0$ for an extremum of $I$ as

$$(F - y'F_{y'})_{x=\bar{x}-0} \cdot \delta\bar{x} + (F_{y'})_{x=\bar{x}-0} \cdot \delta\bar{y}$$

$$- (F - y'F_{y'})_{x=\bar{x}+0} \cdot \delta\bar{x} - (F_{y'})_{x=\bar{x}+0} \cdot \delta\bar{y} = 0$$

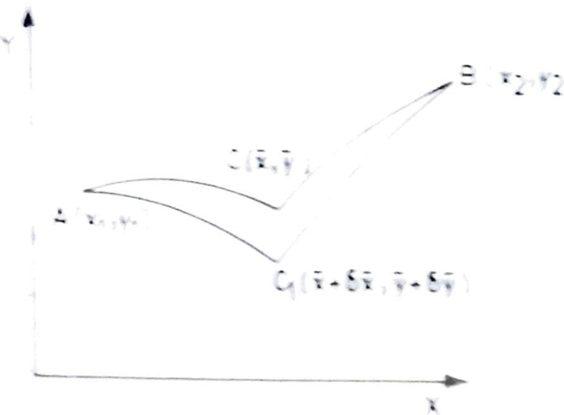


Fig. 2.7 Extremal having one corner point

Since $\delta\bar{x}$ and $\delta\bar{y}$ are independent, we find from above that

$$(F - y'F_{y'})_{x=\bar{x}-0} = (F - y'F_{y'})_{x=\bar{x}+0} \tag{2.35a}$$

and

$$(F_{y'})_{x=\bar{x}-0} = (F_{y'})_{x=\bar{x}+0} \tag{2.35b}$$

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These conditions (known as Weirstrass-Erdmann corner conditions) along with the continuity conditions of the desired extremal determine the coordinates of the corner point.

## 2.5 Diffraction of Light Rays

In Section 2.4, we have seen that the laws of geometrical optics involving reflection or refraction of light are characterized by variational principles. These are based on the assumption that light travels along certain curves (called rays) determined by Fermat's principle (see Example 8 of Chapter 1). But experience shows that while this theory is essentially correct, there are still many cases in which light appears in regions where there are no rays. This discrepancy between geometrical optics and experience accounts for the phenomenon of diffraction. However, geometrical optics can be suitably modified to include this new phenomenon. Such modification consists in introducing new rays, called diffracted rays, and can be formulated in terms of an extension of Fermat's principle (Keller, [9]).

Diffracted rays are generally produced, when a ray hits an edge or a vertex of a cone, or when a ray grazes an interface (between two media). Geometrical optics does not indicate what happens in these cases. At this stage we introduce diffracted rays. Once these rays are introduced, one can define diffracted wave fronts and the phase, or eiconal function by means of them. Let us cite an example in which the diffracted rays cover the shadows of ordinary geometrical optics. Consider an incident ray hitting the edge of a thin screen (Fig. 2.8). The incident ray produces many diffracted rays, travelling in the directions determined by laws of diffraction. This law states that each diffracted ray, which lies in the same medium as the incident ray, makes the same angle with the edge as does the

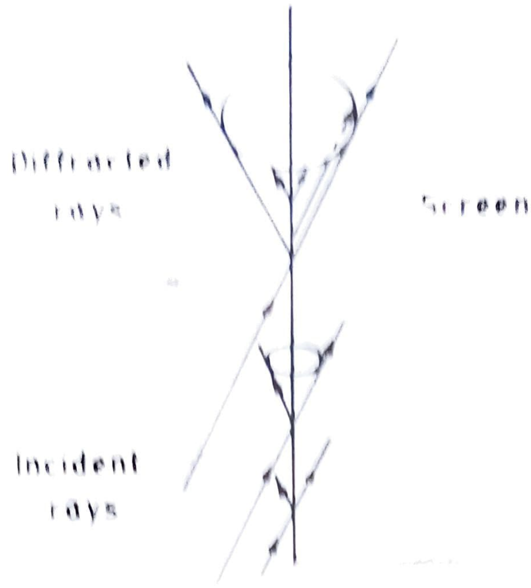


Fig. 2.8 The cone of diffracted rays produced by an incident ray which strikes the edge of a thin screen

incident ray. Further, the incident and diffracted rays lie on opposite sides of the plane, normal to the edge at the point of diffraction. But the diffracted ray need not lie in the same plane, as the incident ray, and the edge. Hence the diffracted rays form the surface of a cone with its vertex at the point of diffraction (as in Fig. 2.8). If a diffracted ray and the incident ray lie in different media, the angle between the diffracted ray and the edge is connected with the angle between the incident ray and the edge by the Snell's law described in Section 2.4. Here also the diffracted ray is not restricted to lie in the same plane as the incident ray and the edge. Thus these diffracted rays also form a conical surface.

When an incident ray hits the vertex of an opaque cone, the diffracted rays emerge from the vertex in all directions as shown in Fig. 2.9.


Fig. 2.9 Diffracted rays propagating in all directions from the tip of a cone.

Finally, Fig. 2.10 illustrates some of the diffracted rays produced when a plane wave hits an opaque convex cylinder. One of the grazing (tangent to the surface) rays is shown. This ray splits, part of it continuing unaffected, and the other part running along the surface. At each point of its path this surface ray sheds a diffracted ray tangent to the path.

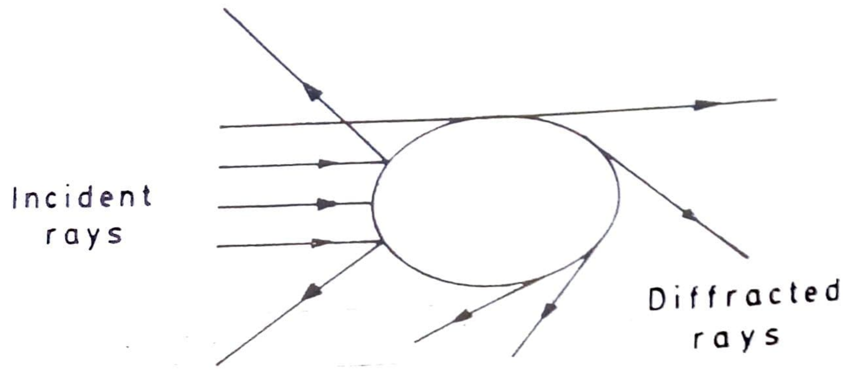


**Fig. 2.10** Diffracted and reflected rays when a plane strikes an opaque convex body at grazing incidence.

We now discuss the generalization of Fermat's variational principle mentioned before. This principle can also be formulated in terms of the index of refraction $n(x)$, which is a positive real function, characterizing the optical behaviour of the medium. The optical length $L$ of any curve connecting two points $P$ and $Q$ is defined as

$$L = \int_P^Q n[x(s)]\, ds,$$

where the parameter $s$ denotes the arc length. The link between this integral and the one stated in the Example 7 of Chapter 1 is that $n = C_0/C$, where $C_0$ and $C$ are the velocities of light in a vacuum and the given medium, respectively. Thus we may restate Fermat's principle as follows: the optical rays connecting $P$ and $Q$ are those curves, joining $P$ and $Q$, which make $L$ stationary (or minimum) in the class $C^0$ of all smooth curves joining $P$ and $Q$. This principle does not apply to a bounded medium or a medium where $n(x)$ is discontinuous. We have, however, seen in Section 2.4, that in the case of reflection or refraction of light at the interface between two media, we have to enlarge the class of curves, so as to include class $C^1$. We can similarly introduce curves belonging to class $C^r$ to account for $r$-tuple reflected or refracted rays. However, this extended class of curves still fails to take account of diffracted rays.

Hence we modify Fermat's principle by introducing additional classes of curves. Let us define a class of curves $D_{rst}$, for each triplet of non-negative integers, $r$, $s$ and $t$. This consists of curves with $r$ smooth arcs on the boundary or discontinuity surfaces, $s$ points on edges of the boundary or discontinuity surfaces, and $t$ points on vertices of these surfaces. Any number of $r$ arcs may be degenerate arcs, i.e., points. To each arc the value of $n$ on one side of the surface is assigned. In the modified Fermat's principle, we define the rays as those curves in each class $D_{rst}$, which make the optical length defined above stationary in $D_{rst}$. Clearly, the class $D_{000}$ is the previously considered class $C^0$ and $D_{r00}$ contains the class $C^1$ and represents singly reflected/refracted rays. With this modified Fermat's principle, we can show by the usual considerations of calculus of variations that it takes care of reflection, refraction and diffraction of light rays.

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*Example 5.* Find the extremals with corner points of the functional

$$I[y(x)] = \int_{x_1}^{x_2} y'^2 (1 - y')^2 \, dx$$

*Solution.* Since the integrand depends on $y'$ only, the extremals are straight lines $y = C_1x + C_2$. Now at a corner point, the conditions (2.35a) and (2.35b) become

$$[y'^2(1 - y')(1 - 3y')]_{x=\bar{x}-0} = [y'^2(1 - y')(1 - 3y')]_{x=\bar{x}+0}$$

$$[y'(1 - y')(1 - 2y')]_{x=\bar{x}-0} = [y'(1 - y')(1 - 2y')]_{x=\bar{x}+0}$$

If we ignore the trivial possibility, $y'(\bar{x} - 0) = y'(\bar{x} + 0)$, we find that the above equations are satisfied for either

$$y'(\bar{x} - 0) = 0, \quad y'(\bar{x} + 0) = 1$$

or

$$y'(\bar{x} - 0) = 1, \quad y'(\bar{x} + 0) = 0$$

Thus the broken line extremals consist of segments of straight lines belonging to the families $y = A_1$ and $y = x + A_2$ as shown in Fig. 2.11.

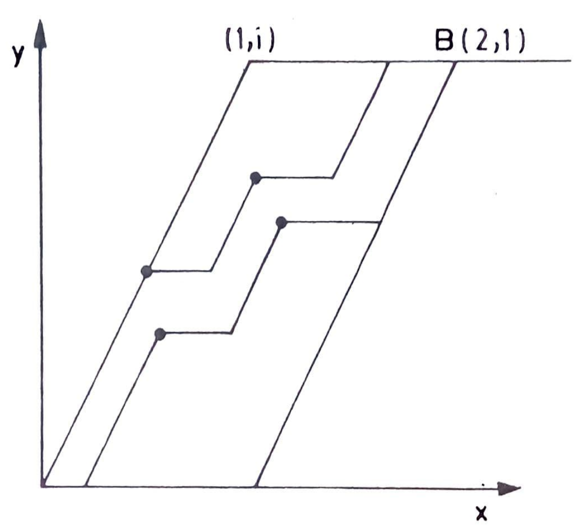


**Fig. 2.11** Extremal having two corner points.

If the integrand in $I[y(x)]$ is denoted by $F$, then

$$F_{y'y'} = 2y'(1 - y')(1 - 2y')$$

which may vanish. This is a possible indication of the existence of a corner point. In fact by du Bois Reymond's theorem (see Section 1.9), the extremal is twice continuously differentiable if $F_{y'y'} \neq 0$.

## PROBLEMS

1. Find the function on which the following functional can be extremized

$$I[y(x)] = \int_0^1 (y''^2 - 2xy)\,dx,\ y(0) = y'(0) = 0,$$

$y(1) = 1/120$ and $y'(1)$ is not given.

Ans. $y = \frac{x^5}{120} + \frac{1}{24}(x^2 - x^3).$

2. Are there any solutions with corner points in the extremum of the functional

$$I[y(x)] = \int_0^{x_1} (y'^4 - 6y'^2)\,dx,\ y(0) = 0,\ y(x_1) = y_1 ?$$

Ans. The polygonal lines passing through the given boundary points consist of rectilinear segments with slopes $\sqrt{3}$ and $-\sqrt{3}$.

3. Determine the stationary function $y(x)$ for the problem

$$\delta\left\{\int_0^1 y'^2\,dx + [y(1)]^2\right\} = 0 \text{ with } y(0) = 1.$$

Ans. $y = 1 - \frac{x}{2}.$

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4. Find the curves on which the following functional can attain an extremum

$$I[y] = \int_0^{10} y'^3\,dx, \quad y(0) = 0, \quad y(10) = 0$$

subject to the condition that the admissible curves cannot pass inside the area bounded by the circle

$(x - 5)^2 + y^2 = 9$

Ans. $y(x) = \pm \frac{3}{4}x$ for $0 \le x \le \frac{16}{5}$

$y(x) = \pm\sqrt{9 - (x - 5)^2}$ for $\frac{16}{5} < x \le \frac{34}{5}$

$y(x) = \pm \frac{3}{4}(x - 10)$ for $\frac{34}{5} < x \le 10.$

5. If $l$ is not pre-assigned, show that the stationary functions corresponding to the problem

$$\delta\int_0^l y'^2\,dx = 0, \quad y(0) = 2, \quad y(l) = \sin l$$

are of the form $y = 2 + 2x\cos l$, where $l$ satisfies

$2 + 2l\cos l - \sin l = 0.$