$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$

$$\Rightarrow \left| \int_E f \right| \leq \int_E |f|.$$

Let $|f|$ be integrable.

Since $f \leq |f|$ f is also integrable.

Hence the integrability $|f|$ implies that of f.

Hence the Proved.

Unit - III

Differentiation and Integration

Differentiation of monotons functions :- impt-I

Let $\mathcal{J}$ be a collection of intervals then we say that $\mathcal{J}$ covers a set E in the sense of vitali, if for each $\epsilon > 0$ and any x in E, there is an interval such that $x \in I$ and $l(I) < \epsilon$.

Note :-

i) $\mathcal{J}$ is called as vitali's cover of the set E.

ii) The intervals may be open, closed (or) half open.

But we do not allow degenerate intervals consisting of only one point.

Vitali's lemma - 1 : - 10 H

Let E be a set of finite outer measure and $\mathcal{I}$ be a collection of intervals that cover E in the sense of Vitali, then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, I_2, \ldots I_N\}$ of intervals in $\mathcal{I}$ such that

$$m^*\left\{E \sim \bigcup_{n=1}^{N} I_n\right\} < \epsilon.$$

Proof : -

It is sufficient to prove the lemma in the case that each interval $\mathcal{I}$ is closed, for otherwise we replace each interval by its closure.

This is possible.

Since the set of end points of $I_1, I_2, \ldots I_N$ has measure zero.

Let "O" be an open set of finite measure containing E.

ie) $E \subset O$ & $mO < \infty$

Since $\mathcal{I}$ is Vitali covering of E, without loss of generality we may assume that each I of $\mathcal{I}$ is contained in O.

ie> $I \subset O \ \forall \ I \in \mathcal{J}$.

Now let us choose a sequence $\{I_n\}$ of disjoint intervals of ~~$\mathcal{J}$~~ $\mathcal{J}$ by induction follows.

Let K_1 be the lub of the lengths intervals in $\mathcal{J}$ which donot meet any of the intervals $I_1, I_2, \dots I_n$.

Let I_1 be any intervals in $\mathcal{J}$.

Since $K_1 < m O < \infty$. $K < \infty$

Let $K_1 = \sup \{ l(I) \}, I \in \mathcal{J}$ & I doesnot meet $I_1 \}$.

Let I_2 be any intervals in ~~$\mathcal{J}$~~ $\mathcal{J}$ which is disjoint from I_1.

Such that $l(I_2) > \frac{1}{2} K_1$.

Let $K_2 = \sup [l(I), I \in \mathcal{J}$ & I doesnot meet I_1 & $I_2]$

clearly $K_2 \leq K_1$ from this way we can choose a disjoint collection of intervals $I_1, I_2, \dots I_n$ in $\mathcal{J}$.

Let K_n be the sup of the length of the intervals of $\mathcal{J}$ that don't meet of the intervals $I_1, I_2, \dots I_n$.

clearly $K_n \leq K_{n-1} \leq \dots K_3 \leq K_2 \leq K_1$.

Since each I is contained in O. We have $K_n \leq \mu(O) < \infty$

Unless $E \subset \bigcup_{i=1}^{n} I_i$

We can find I_{n+1} in of with

$l(I_{n+1}) > \frac{1}{2} K_n \rightarrow (1)$

and I_{n+1} disjoint from $I_1, I_2, \ldots I_n$.

Thus we have a sequence $\{I_n\}$ of a disjoint intervals of $\mathcal{I}$.

Since $\bigcup I_n \subset O$, $m(\bigcup I_n) \leq mO < \infty$

$\sum l(I_n) \leq mO < \infty$.

Hence $\exists$ as integer. N Such that

$$\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5} \rightarrow (2)$$

Let $R = E \sim \bigcup_{n=1}^{N} I_n$

We have to prove

$m^* R < \epsilon$.

Let $x \in R$

Then $x \notin \bigcup_{n=1}^{N} I_n$.

ie) $\bigcup_{n=1}^{N} I_n$ is closed set not containing "x".

We can find interval I in $\mathcal{S}$ of which contains x and $l(I)$ so small that I, doesnot meet any of the interval $I_1, I_2, \ldots, I_n$.

Suppose $I \cap I_i = \phi$ for $i \leq n$. Then we must have

$$l(I) \leq K_n < 2l(I_{n+1})$$

Since $\lim_{n \to \infty} l(I_n) = 0$

$\therefore$ $l(I) = 0$ which is impossible.

$$I \cap I_i \neq \phi \text{ for } i \leq n$$

$\therefore$ ie) I must be meet atleast one of the intervals I_n.

Let n be the smallest integers such that I meet I_n.

We have $n > N$ and $l(I_n) \leq 2l(I_n)$ $\to (2)$.

$$l(I) \leq K_{n-1} \leq 2l(I_n) \to (3)$$

Since $x \in I$ and I meets I_n, the distance from x to the mid point of I_n is atmost.

But $l(I) + \frac{1}{2} l(I_n) \leq 2l(I_n) + \frac{1}{2} l(I_n)$

[by (3)].

Thus $x \in \mathcal{J}_n$ having the same limit points as I_n & $\mathcal{J}$ 5 times the length of I_n and

$$l(\mathcal{J}_n) = 5\, l(I_n) \quad \forall \; n \geq N+1 \rightarrow (4)$$

clearly, $x \in \mathcal{J}_n$

$$R \subset \bigcup_{n=N+1}^{\infty} \mathcal{J}_n \text{ and}$$

$$m^* R \leq \sum_{n=N+1}^{\infty} l(\mathcal{J}_n)$$

$$\leq 5 \sum_{n=N+1}^{\infty} l(I_n) \quad (\text{by } 0)$$

$$\leq \cancel{5}\, {}^{\epsilon}/_{\cancel{5}}$$

$$= \epsilon.$$

Theorem :-

Let f be an increasing real-valued function on the interval $[a,b]$, then f is differentiable almost everywhere. The derivative f' is measurable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

Proof :-

Given that f is an increasing real valued function on the interval $[a,b]$.

To Prove f is differentiable a.e

i.e.) To Prove f is differentiable extend on a set of measure zero.

Let us prove that the sets where any two derivatives are unequal have measure zero.

Let $E = \{x / D^+ f(x) > D^- f(x)\}$.

We shall prove that $m^* E = 0$.

We note that E is the union of the sets of the form $E_{u,v}$.

Now, let $E_{u,v} = \{x : D^+ f(x) > u > v > D^- f(x)\}$

for all rationals u and v.

Claim $E = \bigcup_{u,v} E_{u,v}$

Now $x \in E_{u,v} \Leftrightarrow D^+ f(x) > D^- f(x)$

Then there exist $u, v \in \mathbb{Q}$ such that

$$D^+ f(x) > u > v > D^- f(x)$$

$\Rightarrow x \in E_{u,v}$ for some $u, v \in \mathbb{Q}$.

$$E = \bigcup_{u,v \in \mathbb{Q}} E_{u,v}$$

Hence the claim.

<u>To Prove</u> : $m^* E_{u,v} = 0$.

Suppose $m^* E_{u,v} = s$

let "O" be a open set containing $E_{u,v}$

Such that $m\,O < s+\epsilon$, $h>0$ for each

Point, $x \in E_{u,v} \Rightarrow D^+ f(x) > u > v > D^- f(x)$

Consider $D^- f(x) < v$

$$\Rightarrow \lim_{h \to 0^+} \left[\frac{f(x) - f(x-h)}{h} \right] < v$$

$$\Rightarrow \frac{f(x) - f(x-h)}{h} < v$$

$$\Rightarrow f(x) - f(x-h) < hv.$$

for each point x in $E_{u,v}$ there is an arbitrary small interval $[x-h, x]$ Contained in O Such that,

$$f(x) - f(x-h) < vh \rightarrow (1).$$

The collection of these intervals from a Vitali covering for $E_{u,v}$.

$\therefore$ By Vitali's lemma.

there is an finite collection $\{I_1, I_2, \dots I_n\}$ of closed intervals whose interior cover a Subset A of $E_{u,v}$ of outer measure ~~gr~~ greater ~~to~~ than $s-\epsilon$.

Then Summing eqn (1) over these

intervals, we got

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \sum_{n=1}^{N} h_n$$

$$< v m_0$$

$$< v(s+\epsilon)$$

NHow each point $y \in A$ is the left end point an arbitrarily small interval $(y, y+k)$, since

There is arbitrarily small that

$$f(y+k) - f(y) > uk \rightarrow (3)$$

again by using finite lemma.

We obtain a finite collection $\{J_1, J_2, \dots J_m\}$ of such intervals such that their union contains a subset A of outmeasure greater than $s - 2\epsilon$ than

Summing eqn (3) over these intervals. We get,

$$\sum_{i=1}^{M} [f(y_i + k_i) - f(y_i)] > u \sum_{i=1}^{M} k_i$$

$$> u(s - 2\epsilon) \rightarrow (4)$$

Since each interval J_i is contained in some interval I_n and if we sum over those "i" for which $J_i \subset I_n$. we have

$\sum f(y_i+k_i)-f(y_i) \leq f(x_n)-f(x_n-h_n)$

[$\because$ Since f is increasing]

$$\sum_{n=1}^{N}\left[f(x_n)-f(x_n-h_n)\right] \geq \sum_{i=1}^{N}\left[f(y_i+k_i)-f(y_i)\right]$$

from (2) & (4)

$$v(s+\epsilon) \geq u(s-2\epsilon)$$

Since ϵ is arbitary.

$$vs \geq us$$

$$\Rightarrow v > u \quad \text{But } u > v$$

which a contradiction $u > v$

$$\therefore s=0$$

$$m^* E_{u,v}=0$$

$$m^* E = m^*\left[\bigcup_{u,v\in Q} E_{u,v}\right]=0$$

$$\Rightarrow m^* E = 0.$$

This shows that we can say

$$g(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Then g is defined a.e and f is differential whenever g is finite.

Let $g_n(x) = n\left[f(x+1/n), f(x)\right]$

where the set $f(x)=f(b)$ for $x \geq b$, Then $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \dfrac{f(x+1/n)-f(x)}{1/n}$

$$= \lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n}$$

$$\lim_{n\to\infty} g_n(x) = g(x).$$

$\Rightarrow g_n(x) \to g(x)$ a.e and g is measurable. Since f is increasing, we have $g_n \geq 0$. Hence by fatou's lemma

$$\int_a^b g \leq \underline{\lim} \int_a^b g_n$$

$$= \underline{\lim} \int_a^b n\left[f(x+1/n)-f(x)\right]$$

$$= \underline{\lim}\, n \int_a^b \left[f(x+1/n)-f(x)\right]$$

$$= \underline{\lim}\, n \left\{\int_{a+1/n}^{b+1/n} f(x)\,dx - \int_a^b f(x)\,dx\right\}$$

$$= \underline{\lim}\, n \left[\cancel{\int_{a+1/n}^{b} f(x)\,dx} + \int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f(x)\,dx - \cancel{\int_{a+1/n}^{b} f(x)\,dx}\right]$$

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$$= \underline{\lim}\, n \left[\int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f(x)\,dx \right]$$

$$= \underline{\lim}\, n \left\{ \int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f(x)\,dx \right\}$$

$$= f(b) - \underline{\lim}\, n \int_a^{a+1/n} f(x)\,dx$$

$$\int_a^b g \leq f(b) - f(a)$$

This shows that g is integrable and hence finite a.e.

Thus f is differentiable a.e and $g = f'(x)$.

$$\Rightarrow \int_a^b f'(x)\,dx \leq f(b) - f(a).$$

Hence the proof.

1) Show that $D^+(-f(x)) = -D^+ f(x)$.

Sol! -

$$D^+(-f(x)) = \overline{\lim_{h \to 0^+}} - \frac{[f(x+h) - f(x)]}{h}$$

$$= - \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$= -D^+ f(x).$$

9. If $g(x) = f(-x)$ then $D^+ g(x) = -D_- g(x)$.

Proof :-

Given $f(x) = g(x)$

W.K.T $D^+ f(x) = \overline{\lim_{h \to 0}} \dfrac{f(x+h) - f(x)}{h}$

replace $x = -x$

$$\Rightarrow D^+ f(-x) = \overline{\lim_{h \to 0^+}} \frac{f(-x+h) - f(-x)}{h}$$

$$= \overline{\lim_{h \to 0^+}} \frac{f(-(x-h)) - f(x))}{h}$$

$[f(-x) = g(x)]$

$$= \overline{\lim_{h \to 0^+}} \frac{g(x-h) - g(x)}{h}$$

$$D^+ g(x) = -\underline{\lim_{h \to 0^+}} \frac{g(x) - g(x-h)}{h}$$

$$D^+ g(x) = -D_- g(x)$$

Hence the Proved.

Functions of bounded variations :-

Bounded variation :-

Let f be a real valued function defined on interval $[a, b]$ and

Let $a = x_0 < x_1 < x_2 \dots < x_K = b$ be any subdivision of $[a, b]$

Define $P = \sum_{i=1}^{K} [f(x_i) - f(x_{i-1})]^+$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

Now $t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$

where $r^+ = \begin{cases} r & \text{if } r \geq 0 \\ 0 & \text{if } r \leq 0 \end{cases}$ and

$$r^- = |r| - r^+$$

We have $f(b) - f(a) = p - n$

Clearly $p - n = f(b) - f(a)$

Let $P = \text{Sup } p$

$N = \text{Sup } n$, &

$T = \text{Sup } t$

Where we ~~to~~ take the supremum over all possible subdivisons of $[a, b]$.

clearly, we have $P \leq T \leq P + N$.

An P, N, T are called the +ve, -ve and total variation of f over $[a, b]$.

We sometimes write T_a^b, $T_a^b(f)$ etc., dependence on $[a, b]$ or on the function f.

If $T < \infty$ we say that f is of bounded variation over $[a, b]$.

This notation is sometimes abbreviated by writting $f \in B_V$.

Note :-

If T is the ~~the~~ total variation of f

over $[a,b]$, we denote it as T_a^b (or) $T_a^b(f)$.

Lemma :-

If f is of bounded variation on $[a,b]$ then $T_a^b = P_a^b + N_a^b$ & $f(b) - f(a) = P_a^b - N_a^b$.

Proof :-

For any subdivison of $[a,b]$, consider all subdivision p, n, t

We have $p - n = f(b) - f(a) \rightarrow (1)$

$= n + (f(b) - f(a))$

Taking supremum over all possible subdivision $\phi(a,b)$ we get,

$$P - N = f(b) - f(a) \rightarrow (2)$$

Since $N \leq T < \infty$

$$P \leq N + f(b) - f(a)$$

also $T \geq p + n = n + p$

$$= p - \{f(b) - f(a)\} + p$$

$$T = 2p - (f(b) - f(a))$$

Taking supremum we get,

$$T = 2P - (f(b) - f(a))$$

$$= 2P - (P - N) \quad [\because (2)]$$

$$T \leq 2P - P + N$$

$$T = P + N$$

ie) $T_a^b = P_a^b + N_a^b$

also, (2) $\Rightarrow P_a^b - N_a^b$

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$= f(b) - f(a)$

Hence the proved.

Theorem :-

A function f is f of bounded variation on $[a, b]$ iff f is the difference of 2 monotone real valued function on $[a, b]$.

Proof :-

Assume that f is of bounded variation on $[a, b]$.

We shall P.T f is the difference of 2 monotone real valued functions on $[a, b]$.

[Let $g(x) = P_a^x$ & $h(x) = N_a^x$

where $x \in [a, b]$

Since $0 \leq P_a^x \leq T_a^x \leq T_a^b < \infty$

[$\because$ f is of bounded variation on $[a, b]$]

and $0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty$].

$\therefore$ g and h are real valued monotone increasing function on $[a, b]$

$+ \Rightarrow P_a^b + N_a^b$

But $g(x) - h(x) = P_a^x - N_a^x$

$- \Rightarrow f(b) - f(a)$

where $x \in [a, b]$

Then by lemma

" If f is of bounded variation on $[a, b]$

then $T_a^b = P_a^b + N_a^b$

$f(b) - f(a) = P_a^b - N_a^b$

we have

$g(x) - h(x) = f(x) - f(a) \quad \forall x \in [a, b]$.

$f(x) = g(x) - h(x) + f(a)$

$f(x) = g(x) - \{h(x) - f(x)\}$.

Since h is monotone increasing

h - f(a) is also monotone increasing.

Thus f is expressed as the distance of two monotone real valued function on [a, b].

Conversley,

Assume that f is the difference two monotone real valued functions on [a, b]. Suppose f = g - h on [a, b] where g and h are monotone increasing functions for any Subdivision of [a, b].

We have, $f = \sum |f(x_i) - f(x_{i-1})|$

$g = \sum [g(x_i) - g(x_{i-1})]$

$h = \sum [h(x_i) - h(x_{i-1})]$

$\sum |f(x_i) - f(x_{i-1})| \leq \sum [g(x_i) - g(x_{i-1})] +$
$\sum [h(x_i) - h(x_{i-1})]$

$= g(b) - g(a) + h(b) - h(a)$

Taking supremum over all possible Subdivision on [a, b].

We have, $T_a^b(f) \leq g(b) - g(a) + h(b) - h(a)$

$< \infty$

[g and h are real valued function]

$\therefore$ Hence f is of bounded variation on [a, b].

Hence the proved.

Corollary :-

If f is of bounded variation on [a, b] then $f'(x)$ exist for almost all x in [a, b].

Proof :-

Given that $f \in$ on [a, b] then by above theorem.

f is the difference of two monotone real valued function on [a, b].

ie) $f = g \cdot h$

where g and h are increasing function on [a, b].

Then by the theorem,

"Let f be increasing and valued on [a, b]. Then f is differentiable almost everywhere".

We have g and h are differentiable almost everywhere.

$\Rightarrow g'(x)$ and $h'(x)$ exists for almost all x in $[a,b]$,

$f'(x) = g'(x) - h'(x)$ exists for almost x in $[a,b]$. Differentiation of an integral.

If f is an integrable function defined on $[a,b]$. Then we define its indefinite integral to be the function F defined ∴ on $[a,b]$ by $F(x) = \int_a^x f(t)dt$.

Lemma - 3.6 :-

If f is integrable on $[a,b]$ then the function F define by $F(x) = \int_a^x F(t)dt$ is a continuous function of bounded variation on $[a,b]$.

Proof :-

Given f is integrable on $[a,b]$

$$F(x) = \int_a^x F(t)\,dt$$

To prove $F(x)$ is continuous on $[a,b]$

Let $c \in [a,b]$.

Then by the proposition,

" Let f be a non-negative function which is integrable over a set E. Then for every $\epsilon > 0$ there is $\delta > 0$ such that for every set $A \subset C$ with $mA < \delta$ we have $\int_A f < \epsilon$".

If $A = [c, x]$, we get

Given $\epsilon > 0$ there exist $\delta > 0$ such that $A \subset [a, b]$ with $mA = x - c < \delta$.

Given $\epsilon > 0$ there exist a $\delta > 0$ such that $|x - c| < \delta$.

$$\Rightarrow |F(x) - F(c)| = \left| \int_a^x f(t)\,dt - \int_a^c f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt + \int_c^a f(t)\,dt \right|$$

$$= \left| \int_c^x f(t)\,dt \right|$$

$$|F(x) - F(c)| \leq \int_c^x |f(t)|\,dt$$

$$< \epsilon$$

$\therefore$ F is continuous on $[a, b]$.

To Prove:

F is of bounded variation on $[a, b]$.

Let $a = x_0 < x_1 < x_2 < \ldots < x_K = b$ be a subdivision of $[a, b]$.

Then $t = \sum_{i=1}^{K} |F(x) - F(x_{i-1})|$

$$= \sum_{i=1}^{K} \left| \int_a^{x_i} f(t)\,dt + \int_{x_{i-1}}^{a} f(t)\,dt \right|$$

$$= \sum_{i=1}^{K} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right|$$

$$\leq \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f(t)|\,dt$$

$$= \int_{x_0}^{x_1} |f(t)|\,dt + \int_{x_1}^{x_2} |f(t)|\,dt + \cdots + \int_{x_{k-1}}^{x_k} |f(t)|\,dt$$

$$= \int_{x_0}^{x_k} |f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt$$

$$t \leq \int_a^b |f(t)|\,dt.$$

Taking suprema over all possible subdivison on $[a,b]$,

$$T_a^b \leq \int_a^b |f(t)|\,dt < \infty$$

$$\therefore T_a^b < \infty.$$

f is of bounded variation on $[a,b]$.

Hence the proof.

Lemma - 3.7 :-

If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$. Then $f(t) = 0$ a.e in $[a,b]$.

Proof :-

Given f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$.

To Prove : $f(t) = 0$ a.e on $[a, b]$.

Suppose $f(x) > 0$ on a set E of Positive measure. Then by little wood's first principle there is a closed set F is contained in E with $mF > 0$.

Let $O = (a, b) \cap F$

(Then O is a disjoint union of Countable Collection $\{(a_n, b_n)\}$ of Open intervals).

[$F \subset E$ with $m(E \cap F) < \epsilon$

$0 < m(E) - m(F) < \epsilon$

$0 < mE - \epsilon < mF < mF$]

$mE < mF + \epsilon$

Then either $\int_a^b f \neq 0$ or else $\int_a^b f = 0$

ie) $\int_a^b f(t)\,dt = 0$

$\int_a^b f = 0$ implies $\int_{O \cup f} f = 0$

$\int_O f + \int_f f = 0$

$\Rightarrow \int_O f = -\int_F f \neq 0$ [$\because f > 0$ on E]

But $O = \bigcup_n (a_n, b_n)$ where (a_n, b_n) are disjoint for all ϵ.

[$E = \bigcup E_i$ thm (2 II unit]

$\int_O f = \Sigma \int_{a_n}^{b_n} f \neq 0$. $\int_E f = \Sigma \int_{E_i} f$

$$\int_{a_n}^{b_n} f \neq 0 \text{ for some } n.$$

$$\int_{a_n}^{a} f + \int_{a}^{b_n} f \neq 0$$

$$\int_{a}^{b_n} f - \int_{a}^{a_n} f \neq 0 \text{ for some } n.$$

$\therefore$ Either $\int_a^{b_n} f \neq 0$ (or) $\int_a^{a_n} f \neq 0$ for some n.

$\Rightarrow \int_a^{x} f \neq 0$ for some $x \in [a, b]$.

In any case we see that if f is positive on a set of positive measure.

Then for some $x \in [a, b]$.

We have $\int_a^x f(t)\,dt \neq 0$.

|||ly for $f(x) < 0$ on a set E of positive measure.

Hence the theorem follows by contradiction positive statement.

ie) $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$.

$\Rightarrow f(t) = 0$ a.e on $[a, b]$.

Lemma - 3.8 :-

If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$ then

$F'(x) = f(x)$ for almost all x in $[a,b]$.

Proof:-

Given that f is bounded and measurable on $[a,b]$ and

$$F(x) = \int_a^x f(t)\,dt + F(a)$$

To prove

$$F'(x) = f(x) \text{ a.e on } [a,b]$$

By lemma, " If f is integrable on $[a,b]$ then the function F defined by $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on $[a,b]$".

We have F is of Bounded variation over $[a,b]$ and so $F'(x)$ exists for almost all x in $[a,b]$ (by corr 3.5).

Since f is bounded.

Let $|f| \leq k$

and Let $f_n(x) = \dfrac{F(x+h) - F(x)}{h}$ where $h = 1/n$

$$f_n(x) = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt + F(a) - F(a)\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

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$$|f_n(x)| \le \frac{1}{h} \int_x^{x+h} |f(t)|\,dt$$

$$= \frac{1}{h} \cdot k\,[x+h-x]$$

$$= k.$$

i.e., $|f_n(x)| \le k \quad \forall n$ & $x \in [a,b]$.

Also $\lim_{n\to\infty} f_n(x) = \lim_{h\to 0} \left[\frac{F(x+h)-F(x)}{h} \right]$

$$\lim_{n\to\infty} f_n(x) = F'(x) \text{ a.e on } [a,b].$$

$\{ \because F'(x)$ exists a.e on $[a,b] \}$

Let us $\{f_n\}$ is a sequence of measurable continuous con satisfying in the hypothesis of the bounded convergence theorem.

Let $c \in [a,b]$.

Thus $\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c f_n(x)\,dx$

$$= \lim_{h\to 0} \int_a^c \left(\frac{f(x+h)-f(x)}{h} \right) dx$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_a^c F(x+h)\,dx - \int_a^c F(x)\,dx \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_{a+h}^{c+h} F(x)\,dx - \int_{a}^{a+h} F(x)\,dx - \int_{a+h}^{c} F(x)\,dx \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_{c}^{c+h} F(x)\,dx - \int_{a}^{a+h} F(x)\,dx \right]$$

$$= \lim_{h \to 0} \left[\frac{G(c+h) - G(c)}{h} \right] - \lim_{h \to 0} \left[\frac{G(a+h) - G(a)}{h} \right]$$

$$\therefore \int_a^c F'(x)\,dx = G'(c) - G'(a) \quad [\because G' = F]$$

$$= \int_a^c f(x)\,dx + F(a) - F(a)$$

$$\int_a^c F'(x)\,dx = \int_a^c f(x)\,dx$$

Since F is continuous, hence $c \in [a, b]$

$$\Rightarrow \int_a^c F'(x)\,dx - \int_a^c f(x)\,dx = 0 \quad \text{on } [a, b]$$

$\Rightarrow F'(x) - f(x) = 0$ a.e by lemma.

$\Rightarrow F'(x) = f(x)$ a.e.

Theorem :-

Let F be a integrable function $[a, b]$ and, suppose that $F(x) = \int_a^x f(t)\,dt + F(a)$. then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Proof :-

without loss of generality.

We may assume that $f \geq 0$

Let f_n be defined by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n \end{cases}$$

Then $f - f_n \geq 0$ and so

$G_n(x) = \int_a^x f - f_n$ is an increasing

function of x and $G_n'(x)$ exists a.e

and $G_n'(x) \geq 0$.

Now by lemma (3.8).

$$\frac{d}{dx} \int_a^x f_n = f_n(x) \text{ a.e}$$

$$G_n'(x) = (f - f_n)(x)$$

ie) $\frac{d}{dx} G_n(x) = (f - f_n)(x)$

$$f_n(x) = f - \frac{d}{dx} G_n(x)$$

$$f_n(x) = f - \frac{d}{dx} \int_a^x f(x)\,dx + \frac{d}{dx} \int_a^x f_n(x)$$

$$= f - f + \frac{d}{dx} \int_a^x f_n$$

$$\therefore f_n(x) = \frac{d}{dx} \int_a^x f_n \text{ a.e}$$

and so

$$F'(x) = \frac{d}{dx} G_n + \frac{d}{dx} \int_a^x f_n.$$

$F'(x) \geq f_n(x)$ a.e

Since n is arbitrary $F'(x) \geq f(x)$ a.e

Consequently,

$$\int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f'(x)\,dx \leq f(b) - f(a) \quad \text{[by thm (3)]}$$

$$\int_a^b F'(x)\,dx < F(b) - F(a)$$

$$\therefore \int_a^b (F'(x) - f(x))\,dx = 0$$

Since $F'(x) - f(x) \geq 0$ this implies that

$F'(x) - f(x) = 0$ a.e and so

$F'(x) = f(x)$ a.e

3.4. Absolute Continuity.

Definition:-

A real valued function f defined on $[a,b]$ is said to be absolutely continuous on $[a,b]$ if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon.$$

for every finite collection $\{(x_i, x_i')\}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Note :-

1. An absolutely continuous function is continuous.

2. Every indefinite integral is absolutely continuous.

3. The sum and difference of two absolutely continuous function is absolutely continuous.

Lemma - 13.10 :-

(7) If f is absolutely continuous on $[a, b]$. Then it is of bounded variation on $[a, b]$.

Proof :-

Given that f is abou absolutely continuous on $[a, b]$.

$\therefore$ for $\epsilon = 1 > 0$, there exists $\delta > 0$.

Such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < 1 \rightarrow (1)$$

for all finite collection $\{(x_i, x_i')\}$ of non-over happing subintervals of $[a, b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

Then any subdivision of $[a, b]$ can be split into a set of intervals each of total length less than δ where K is the largest integer.

Such that $k < 1 + \frac{(b-a)}{\delta}$.

Divide $[a, b]$ by means of Points

$$a = x_0 < x_1 < \dots < x_k = b$$

with $x_j - x_{j-1} < \delta$ for $i = 1, 2, \dots k$.

for every finite collection $\{(x_i, x_i')\}$ of non over lapping Subintervals in $[x_{j-1}, x_j]$

We have

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < 1.$$

$$\sum | f(x_i) - f(x_{i-1}) | < 1.$$

$$T_{(x_{i-1})}^{(x_i)} < 1 \quad i = 1, 2, \dots k.$$

$$T_a^b = \sum_{j=1}^{k} T_{(x_{j-1})}^{(x_j)}$$

$$= T_{x_0}^{x_1} + T_{x_1}^{x_2} + \dots + T_{x_{k-1}}^{k}$$

$$< 1 + 1 + \dots + 1 \quad (k \text{ times})$$

ie) $T_a^b < k$, but $k < 1 + \frac{b-a}{\delta} < \infty$

$$\therefore \boxed{T_a^b (f) < \infty}$$

Hence f is of bounded variation on $[a, b]$.

Corollary 3.11 :-

If f is absolutely Continuous, then on f

has a derivative almost everywhere on [a, b].

Proof : -

Since f is absolutely continuous on [a, b] by the above lemma.

$f \in$ Bounded variation on [a, b].

Then by corollary,

If f is of bounded variation on [a, b]. Then $f'(x)$ is exists for almost all x in [a, b].

We have $f'(x)$ exist almost everywhere on [a, b].

Hence f is absolutely continuous on [a, b] implies $f'(x)$ exists almost everywhere on [a, b].

Lemma - 3.12 : - 10M Q.P.

To Prove If f is abosolutely continuous on [a, b] and $f'(x) = 0$ a.e. Then f is constant.

Proof : -

To prove f is constant on [a, b]. We have to porove that $f(a) = f(c)$ for any $c \in [a, b]$.

Let $E \subset (a, c)$ be the set of measure $c - a$ in which $f'(x) = 0$.

ie) $E = \{x \in (a, c) / f'(x) = 0\}$ and $mE = c - a$

let ϵ & η be arbitrary positive numbers.

Let $x \in E$ then $f'(x) = 0$.

$\therefore$ There exists an arbitrarily small interval

$$[x, x+h] \subset [a, c]$$

such that $|f(x+h) - f(x)| < \eta h \rightarrow (1)$

W.K.T.

The collection of intervals

$\{[x, x+h], [x \in E]\}$ is a Vitali's lemma cover for E.

$\therefore$ By Vitali's lemma it is always possible to choose a finite collection of non-over lapping intervals say $\{[x_k, y_k]\}$ $k = 1, 2, \ldots n$ from the Vitali's cover.

These finite intervals cover E except for a set of measure less than δ where δ is the positive number.

Corresponding to ϵ in the def of the absolutely continuous of f.

If we label the x_k so that

$$x_k \leq x_{k+1}$$

We have

$$y_0 = a \leq x_1 \leq y_1 \leq x_2 < \ldots \leq y_n \leq c = x_{n+1}.$$

and

$$\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta.$$

Now,

$$\sum_{K=1}^{n} |f(y_K) - f(x_K)| \leq \eta \sum_{K=0}^{n} (y_K - x_K) < \eta (c-a)$$

By the absolutely continuous of f.

$$\sum_{K=0}^{n} |f(x_{K+1}) - f(y_K)| < \xi.$$

Thus $$|f(c) - f(a)| = \left| \sum_{K=0}^{n} [f(x_{K+1}) - f(y_K)] + \sum_{K=1}^{n} [f(y_K) - f(x_K)] \right|.$$

$$\leq \xi + \eta (c-a).$$

Since ξ & η are arbitrary positive number.

$$|f(c) - f(a)| = 0$$

$f(a) = f(c)$ for any $c \in [a, b]$.

$\therefore f$ is constant.

Theorem - 3.13 :-

A function F is an indefinite integral iff it is absolutely continuous.

Proof:-

Assume that f is absolutely continuous.

To prove F is an indefinite integral we have to prove that

$$F(x) = \int_a^x F'(t)\,dt + F(a).$$

Since F is absolutely continuous, F is of Bounded variation on [a, b].

$\therefore$ F(x) can be written as the difference two increasing functions on [a, b]. i.e) $F(x) = F_1(x) - F_2(x)$.

where F_1 & F_2 are two monotone increasing functions on [a, b].

$\therefore$ F_1 & F_2 are differentiable a.e.

$$F'(x) = F_1'(x) - F_2'(x)$$

$$\Rightarrow |F'(x)| \leq |F_1'(x) - F_2'(x)|.$$

Assume that F_1 & F_2 are non-negative.

$$|F'(x)| \leq F_1'(x) + F_2'(x)$$

$$\int_a^b |F'(x)| dx \leq \int_a^b F_1'(x) dx + \int_a^b F_2'(x) dx$$

$$\leq F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

$$< \infty$$

$\therefore$ F'(x) is integrable over [a, b].

Consider the function G(x) defined by

$$G(x) = \int_a^x F'(t)\, dt$$

Then G is absolutely continuous.

$\therefore$ The function $f = F - G$ is also absolutely continuous.

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Then by the thm," Let f be an integrable function on $[a, b]$ and suppose that $F(x) = \int_a^x f(t)\,dt + F(a)$. Then

$F'(x) = f(x)$ a.e on $[a, b]$".

We have

$$f(x) = \int_a^x F'(t)\,dt + F(a) - \int_a^x F'(t)\,dt$$

$$f(x) = f(a)$$

$\therefore f$ is constant.

Thus $f = F - a$

$$f(a) = F(x) - G(x)$$

$$f(a) + \int_a^x F'(t)\,dt = F(x).$$

F is indefinite integral.

Conversely, Assume that $F(x)$ is an indefinite integral of $f(x)$ define on $[a, b]$.

i.e) $F(x) = \int_a^x f(t)\,dt + F(a)$ for every $x \in [a, b]$.

<u>To Prove</u>:-

$F(x)$ is absolutely continuous. Since F is an indefinite integral.

F is the integrable on $[a, b]$.

Then by the preposition : 14.

" Let f be a non-negative function

which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$, with $mA < \delta$. We have $\int_A f < \epsilon$".

We have given $\epsilon > 0$ there exists $\delta > 0$ such that for every set $A \subset [a, b]$ with $mA < \delta$ and $\int_A |f| < \epsilon \rightarrow (1)$.

Choose 2n real numbers.

$x_1, x_2, \ldots x_n, x_1', x_2' \ldots x_n'$ with

$x_1 < x_1' \leq x_2 < x_2' \leq \ldots \leq x_n < x_n'$.

Thus for a finite collection $\{x_i, x_i'\}$ of pairwise disjoint open intervals in $[a, b]$ with

$$\Sigma |x_i' - x_i| < \delta$$

we have $\sum_{i=1}^{N} \left| \int_{x_i}^{x_i'} f(t)\,dt \right| \leq \sum_{i=1}^{N} \int_{x_i}^{x_i'} |f(t)|\,dt < \epsilon$

$$\Rightarrow \sum_{i=1}^{N} \left| \int_{x_i}^{x_i'} f(t)\,dt \right| = \sum_{i=1}^{N} \left| \int_a^{x_i'} f(t)\,dt - \int_a^{x_i} f(t)\,dt \right| < \epsilon$$

$$\sum_{i=1}^{N} |F(x_i') - F(x_i)| < \epsilon$$

$\Rightarrow$ F is can absolutely continuous.

— X —.